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Algebraic Analysis of the Module Associated to Biregular Functions

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OUTLINE

1. Regularity: **left** or **right**?

2. Some experiments with CoCoA

3. Compatibility conditions and free resolution

4. Non commutative syzygies: a question...

5. Conclusions

REGULARITY

Quaternions : $\mathbb{H}^{2n} = \{(q_1, \ldots, q_n, p_1, \ldots, p_n)\}$

$\mathbf{q}_t = x_{t0} + \mathbf{i}x_{t1} + \mathbf{j}x_{t2} + \mathbf{k}x_{t3}$ $\mathbf{p}_t = y_{t0} + \mathbf{i}y_{t1} + \mathbf{j}y_{t2} + \mathbf{k}y_{t3}$

LEFT Cauchy-Fueter operator $t = 1 \ldots n$

$$D_{\ell_t} = \frac{\partial}{\partial x_{t0}} + \mathbf{i}\frac{\partial}{\partial x_{t1}} + \mathbf{j}\frac{\partial}{\partial x_{t2}} + \mathbf{k}\frac{\partial}{\partial x_{t3}}$$

RIGHT Cauchy-Fueter operator

$$D_{r_t} = \frac{\partial}{\partial y_{t0}} + \frac{\partial}{\partial y_{t1}}\mathbf{i} + \frac{\partial}{\partial y_{t2}}\mathbf{j} + \frac{\partial}{\partial y_{t3}}\mathbf{k}$$

BIREGULAR FUNCTIONS

differentiable functions $\quad f : \mathbb{H}^{2n} \longrightarrow \mathbb{H}$

f is **biregular** on $\Omega \subseteq \mathbb{H}^{2n}$ if it satisfies the system of equations

$$\begin{cases} D_{\ell_1} f(\mathbf{q}, \mathbf{p}) & = & 0 \\ & \cdots & \\ D_{\ell_n} f(\mathbf{q}, \mathbf{p}) & = & 0 \\ D_{r_1} f(\mathbf{q}, \mathbf{p}) & = & 0 \\ & \cdots & \\ D_{r_n} f(\mathbf{q}, \mathbf{p}) & = & 0 \end{cases} \qquad \text{for each}(\mathbf{q}, \mathbf{p}) \in \Omega$$

BIREGULAR FUNCTIONS

differentiable functions $\quad f : \mathbb{H}^{2n} \longrightarrow \mathbb{H}$

consider the **inhomogeneous** system of PDEs - with data g_{ℓ_t}, g_{r_t}

$$\begin{cases} D_{\ell_1} f(\mathbf{q}, \mathbf{p}) & = & g_{\ell_1} \\ & \dots & \\ D_{\ell_n} f(\mathbf{q}, \mathbf{p}) & = & g_{\ell_n} \\ D_{r_1} f(\mathbf{q}, \mathbf{p}) & = & g_{r_1} \\ & \dots & \\ D_{r_n} f(\mathbf{q}, \mathbf{p}) & = & g_{r_n} \end{cases} \qquad \text{for each} (\mathbf{q}, \mathbf{p}) \in \Omega$$

ALGEBRAIC ANALYSIS

symbols $\longrightarrow$ matrices with entries in $R = \mathbb{C}[x_{10}, \ldots, y_{n3}]$

$$\begin{bmatrix} \partial_{x_{t0}} & -\partial_{x_{t1}} & -\partial_{x_{t2}} & -\partial_{x_{t3}} \\ \partial_{x_{t1}} & \partial_{x_{t0}} & -\partial_{x_{t3}} & \partial_{x_{t2}} \\ \partial_{x_{t2}} & \partial_{x_{t3}} & \partial_{x_{t0}} & -\partial_{x_{t1}} \\ \partial_{x_{t3}} & -\partial_{x_{t2}} & \partial_{x_{t1}} & \partial_{x_{t0}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0,$$

$$\ldots$$

$$\begin{bmatrix} \partial_{y_{t0}} & -\partial_{y_{t1}} & -\partial_{y_{t2}} & -\partial_{y_{t3}} \\ \partial_{y_{t1}} & \partial_{y_{t0}} & \partial_{y_{t3}} & -\partial_{y_{t2}} \\ \partial_{y_{t2}} & -\partial_{y_{t3}} & \partial_{y_{t0}} & \partial_{y_{t1}} \\ \partial_{y_{t3}} & \partial_{y_{t2}} & -\partial_{y_{t1}} & \partial_{y_{t0}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0$$

ALGEBRAIC ANALYSIS

compact form of the double Cauchy-Fueter system: $P(D)\vec{f} = \vec{g}$

we study algebraic properties of R^4/M, where $M = \langle$ rows of $P \rangle$

$\longrightarrow$ **compatibility conditions of the system = syzygies of** M

More in general we are interested in the free resolution:

$$0 \longrightarrow R^{\beta_s} \longrightarrow R^{\beta_{s-1}} \longrightarrow \dots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow R^4/M \longrightarrow 0$$

EXAMPLES WITH CoCoA

It is time for a **DEMO**!

FREE RESOLUTION

$n \geq 3$

- The **length** of the resolution is $4n - 2$

- **Betti numbers**: $\ldots \longrightarrow R^{\beta_d} \longrightarrow \ldots \longrightarrow R^{4n} \longrightarrow R^4 \longrightarrow 0$

$$\beta_d = 4n^2 \sum_{i+j=d} \binom{2n-1}{i}\binom{2n-1}{j}\frac{ij+1-d}{ij+1+d}, \quad d > 1.$$

- The resolution becomes **linear** at the step $2n + 2$

INGREDIENTS

- $\mathrm{LT}(M) = \mathrm{LT}(M_\ell) \oplus \mathrm{LT}(M_r) \subseteq R^4$

- $\mathrm{LT}(M) = \Delta_4(I) = Ie_1 \oplus Ie_2 \oplus Ie_3 \oplus Ie_4$, $i = 1, 2$

- $I = I_\ell \oplus I_r \quad \Rightarrow \quad \boxed{R/I \cong R/I_\ell \otimes R/I_r}$

- **Hilbert** Series $\mathcal{H}_{R^4/M}(t) = 4\frac{(1+(n-1)t)^2}{(1-t)^{4n+2}}$

- A minimal **regular sequence** for M has length $4n - 2$

TENSOR PRODUCT

WRONG!

Left $$0 \longrightarrow R^{\mathbf{4}}(-4) \longrightarrow R^{\mathbf{8}}(-3) \longrightarrow R^{\mathbf{8}}(-1) \longrightarrow R^{\mathbf{4}} \longrightarrow 0$$

$$\bigotimes$$

Right $$0 \longrightarrow R^{\mathbf{4}}(-4) \longrightarrow R^{\mathbf{8}}(-3) \longrightarrow R^{\mathbf{8}}(-1) \longrightarrow R^{\mathbf{4}} \longrightarrow 0$$

$$\Downarrow$$

$$\ldots \longrightarrow R^{\mathbf{160}}(-4) \longrightarrow R^{\mathbf{64}}(-2) \oplus R^{\mathbf{64}}(-3) \longrightarrow R^{\mathbf{64}}(-1) \longrightarrow R^{\mathbf{16}} \longrightarrow 0$$

Left $0 \longrightarrow R^{\mathbf{1}}(-4) \longrightarrow R^{\mathbf{2}}(-3) \longrightarrow R^{\mathbf{2}}(-1) \longrightarrow R^{\mathbf{1}} \longrightarrow 0$

$$\bigotimes$$

Right $0 \longrightarrow R^{\mathbf{1}}(-4) \longrightarrow R^{\mathbf{2}}(-3) \longrightarrow R^{\mathbf{2}}(-1) \longrightarrow R^{\mathbf{1}} \longrightarrow 0$

$$\Downarrow$$

$$\ldots \longrightarrow R^{\mathbf{10}}(-4) \longrightarrow R^{\mathbf{4}}(-2) \oplus R^{\mathbf{4}}(-3) \longrightarrow R^{\mathbf{4}}(-1) \longrightarrow R^{\mathbf{1}} \longrightarrow 0$$

TENSOR PRODUCT

CORRECT!

Left $0 \longrightarrow R^{\mathbf{1}}(-4) \longrightarrow R^{\mathbf{2}}(-3) \longrightarrow R^{\mathbf{2}}(-1) \longrightarrow R^{\mathbf{1}} \longrightarrow 0$

$$\bigotimes$$

Right $0 \longrightarrow R^{\mathbf{1}}(-4) \longrightarrow R^{\mathbf{2}}(-3) \longrightarrow R^{\mathbf{2}}(-1) \longrightarrow R^{\mathbf{1}} \longrightarrow 0$

$$\Downarrow \text{ times } 4$$

$$\ldots \longrightarrow R^{\mathbf{40}}(-4) \longrightarrow R^{\mathbf{16}}(-2) \oplus R^{\mathbf{16}}(-3) \longrightarrow R^{\mathbf{16}}(-1) \longrightarrow R^{\mathbf{4}} \longrightarrow 0$$

BIREGULAR FUNCTIONS

differentiable functions $\quad f : \mathbb{H}^{2n} \longrightarrow \mathbb{H}$

consider the **inhomogeneous** system of PDEs - with data g_{ℓ_t}, g_{r_t}

$$\begin{cases} D_{\ell_1} f(\mathbf{q}, \mathbf{p}) & = & g_{\ell_1} \\ & \cdots & \\ D_{\ell_n} f(\mathbf{q}, \mathbf{p}) & = & g_{\ell_n} \\ D_{r_1} f(\mathbf{q}, \mathbf{p}) & = & g_{r_1} \\ & \cdots & \\ D_{r_n} f(\mathbf{q}, \mathbf{p}) & = & g_{r_n} \end{cases} \qquad \text{for each} (\mathbf{q}, \mathbf{p}) \in \Omega$$

COMPATIBILITY CONDITIONS

- **Koszul** type: n^2 $\quad$ $\boxed{D_{\ell i} g_{rj} = D_{rj} g_{\ell i}}$ $\quad$ $i, j \in \{1, \ldots, n\}$

- **Radial** type: $4\binom{n}{2} + 4\binom{n}{3}$ $\quad$ $i, j \in \{1, \ldots, n\}, i \neq j$

$$D_{\ell i} D_{\ell j} g_{\ell j} = D_{\ell j}^2 g_{\ell i}$$

$$D_{\ell i} D_{\ell j} g_{\ell k} + D_{\ell j} D_{\ell i} g_{\ell k} = D_{\ell k} D_{\ell i} g_{\ell j} + D_{\ell k} D_{\ell j} g_{\ell i}$$

- **Exceptional**: $4\binom{n}{3}$ $\quad$ $1 \leq i < j < k \leq n$

$$(D'_{\ell i} D_{\ell j} - D'_{\ell j} D_{\ell i}) g_{\ell k} + (D'_{\ell j} D_{\ell k} - D'_{\ell k} D_{\ell j}) g_{\ell i} + (D'_{\ell k} D_{\ell i} - D'_{\ell i} D_{\ell k}) g_{\ell j} = 0$$

$$(D''_{\ell i} D_{\ell j} - D''_{\ell j} D_{\ell i}) g_{\ell k} + (D''_{\ell j} D_{\ell k} - D''_{\ell k} D_{\ell j}) g_{\ell i} + (D''_{\ell k} D_{\ell i} - D''_{\ell i} D_{\ell k}) g_{\ell j} = 0$$

COMPATIBILITY CONDITIONS

- **Koszul** type: n^2 $\boxed{D_{\ell i} g_{rj} = D_{rj} g_{\ell i}}$ $i, j \in \{1, \dots, n\}$

- **Radial** type: $4\binom{n}{2} + 4\binom{n}{3}$ $i, j \in \{1, \dots, n\}, i \neq j$

$$D_{ri} D_{rj} g_{rj} = D_{rj}^2 g_{ri}$$

$$D_{ri} D_{rj} g_{rk} + D_{rj} D_{ri} g_{rk} = D_{rk} D_{ri} g_{rj} + D_{rk} D_{rj} g_{ri}$$

- **Exceptional**: $4\binom{n}{3}$ $1 \leq i < j < k \leq n$

$$(D'_{ri} D_{rj} - D'_{rj} D_{ri}) g_{rk} + (D'_{rj} D_{rk} - D'_{rk} D_{rj}) g_{ri} + (D'_{rk} D_{ri} - D'_{ri} D_{rk}) g_{rj} = 0$$

$$(D''_{ri} D_{rj} - D''_{rj} D_{ri}) g_{rk} + (D''_{rj} D_{rk} - D''_{rk} D_{rj}) g_{ri} + (D''_{rk} D_{ri} - D''_{ri} D_{rk}) g_{rj} = 0$$

INGREDIENTS

• An element of the Gröbner Basis of M involves at most 2 quaternionic indices

• A syzygy involves it at most 4 indices, actually **3**

• **Lemma**:

$$\begin{array}{ll} A_1, \ldots, A_n, B \text{ operators} & \\ \mathrm{Syz}(A_1, \ldots, A_n) & \Rightarrow \quad \mathrm{Syz}(A_1, \ldots, A_n, B) \\ A_i B = B A_i & \end{array}$$

A QUESTION

Suppose we have 2 sets of symbols $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ in a ring $\mathcal{R}$

Suppose we know:

- $A_i B_j = B_j A_i$
- $\mathrm{Syz}(A_1, \ldots, A_n) \subset \mathcal{R}^n$
- $\mathrm{Syz}(B_1, \ldots, B_m) \subset \mathcal{R}^m$

Question: Can we describe $\mathrm{Syz}(A_1, \ldots, A_n, B_1, \ldots, B_m)$?

COUNTEREXAMPLE

$m = n = 2$

$\mathcal{R} = \langle A_1, A_2, B_1, B_2 \rangle$ such that $A_1^2 = A_2^2 = \mathbf{1}$ and $B_1^2 = B_2^2 = -\mathbf{1}$

$A_i B_j = B_j A_i$, $i, j = 1, 2$

$\Rightarrow \mathrm{Syz}(A_1, A_2) = \langle (A_1, -A_2) \rangle \qquad \mathrm{Syz}(B_1, B_2) = \langle (B_1, -B_2) \rangle$

is it enough to describe $\mathrm{Syz}(A_1, A_2, B_1, B_2)$?

Koszul: $(B_1, 0, -A_1, 0)$, $(B_2, 0, 0, -A_2)$, $(0, B_1, -A_2, 0)$, $(0, B_2, 0, -A_2)$
Expected: $(A_1, -A_2, 0, 0)$, $(0, 0, B_1, -B_2)$

- $(A_1, 0, B_1, 0)$ and $(0, A_2, 0, B_2)$ are syzygies too!

CONCLUSIONS

1. Gröbner Basis theory (and CoCoA) allowed to perform the algebraic analysis of the module associated to biregular functions.

2. Key ingredient: $\mathrm{LT}(M) = \Delta_4(I) \;\rightarrow$ apply to Dirac system

3. Syzygies for left operators and right operators "mix well"

4. Problem: describe when they DO NOT mix well...