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Algebraic Analysis of the System Associated to Rarita–Schwinger operators

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OUTLINE

1. Rarita–Scwhinger operator in *real dimension* 3

2. The **complex** associated to 1, 2 and more operators

3. Exactness, compatibility conditions, **algebraic properties**

4. Rarita–Scwhinger in *real dimension* 4 : some **experiments**

DEFINITION dimension 3

(ρ_m, V_m) irreducible representation of $SU(2)$, $(\rho_2, V_2) \simeq (ad, \mathbb{R}^3 \otimes \mathbb{C})$,

$$(\rho_m, V_m) \otimes (\rho_2, V_2) \simeq (\rho_{m+2}, V_{m+2}) \oplus (\rho_m, V_m) \oplus (\rho_{m-2}, V_{m-2})$$

$$\nabla^S : \Gamma(S_m) \longrightarrow \Gamma(S_m \otimes T^*(M))$$

$\Gamma(S_m \otimes T^* M) \simeq \Gamma(S_{m+2}) \oplus \Gamma(S_m) \oplus \Gamma(S_{m-2})$ $\quad \pi_+, \pi_0, \pi_-$ projections

$$RS := \pi_0 \circ \nabla^S : \Gamma(S_3) \longrightarrow \Gamma(S_3)$$

we need now local coordinates...

FLAT REPRESENTATION

[Y.Homma using mathematica]

$\{e_i\}_{i=1..3}$ a basis of $\mathbb{R}^3$ $\{z^k\}_{k=0..m}$ a basis of V_m

$D^m = \sum_{i=1}^3 \rho_m^0(e_i) \cdot \nabla_{e_i} : \Gamma(S_m) \longrightarrow \Gamma(S_m)$

$$\begin{cases} \rho_m^0(\frac{e_1}{2})z^k & = & i(k - \frac{m}{2})z^k \\ \rho_m^0(\frac{e_2}{2} + i\frac{e_3}{2})z^k & = & -kz^{k-1} \\ \rho_m^0(\frac{e_2}{2} - i\frac{e_3}{2})z^k & = & (m-k)z^{k+1}. \end{cases}$$

FLAT REPRESENTATION

[Y.Homma using mathematica]

$\{e_i\}_{i=1..3}$ a basis of $\mathbb{R}^3$ $\{z^k\}_{k=0..3}$ a basis of V_3

$RS = \sum_{i=1}^{3} \rho_3^0(e_i) \cdot \nabla_{e_i} : \Gamma(S_3) \longrightarrow \Gamma(S_3)$

$$\begin{cases} \rho_3^0(\frac{e_1}{2})z^k & = & i(k - \frac{m}{2})z^k \\ \rho_3^0(\frac{e_2}{2} + i\frac{e_3}{2})z^k & = & -kz^{k-1} \\ \rho_3^0(\frac{e_2}{2} - i\frac{e_3}{2})z^k & = & (m-k)z^{k+1}. \end{cases}$$

Rarita–Schwinger System

differentiable functions $f, g_1, \ldots, g_n : U \subseteq (\mathbb{R}^3)^n \longrightarrow V_3$

consider the **inhomogeneous** system of PDEs - with data $g_1 \ldots g_n$

$$\begin{cases} RS_1 f & = & g_1 \\ & \cdots & \\ RS_n f & = & g_n \end{cases}$$

It can be written in the form $P(D)f = g$ where

P is a matrix of polynomials in $R := \mathbb{C}[x_i^t]_{i=1..3}^{t=1..n}$,
$D = (-i\frac{\partial}{\partial x_1^1}, \ldots, -i\frac{\partial}{\partial x_3^n})$.

ALGEBRAIC ANALYSIS

compact form of the system: $P(D)f = g$

we study algebraic properties of R^s/M, where $M = \langle$ rows of $P \rangle$

$\longrightarrow$ **compatibility conditions of the system = syzygies of M**

More in general we are interested in the free resolution:

$$0 \longrightarrow R^{\beta_s} \longrightarrow R^{\beta_{s-1}} \longrightarrow \ldots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow R^4/M \longrightarrow 0$$

Matrix Representation

(with opportune coordinates):

$$\text{Cauchy}-\text{Fueter}: \begin{bmatrix} \partial_{x_{t0}} & -\partial_{x_{t1}} & -\partial_{x_{t2}} & -\partial_{x_{t3}} \\ \partial_{x_{t1}} & \partial_{x_{t0}} & -\partial_{x_{t3}} & \partial_{x_{t2}} \\ \partial_{x_{t2}} & \partial_{x_{t3}} & \partial_{x_{t0}} & -\partial_{x_{t1}} \\ \partial_{x_{t3}} & -\partial_{x_{t2}} & \partial_{x_{t1}} & \partial_{x_{t0}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0,$$

$$\text{Rarita}-\text{Scwhinger}: \begin{bmatrix} \partial_{X_{t1}} & \partial_{X_{t3}} & 0 & 0 \\ \partial_{X_{t2}} & \partial_{X_{t1}} & 2\partial_{X_{t3}} & 0 \\ 0 & 2\partial_{X_{t2}} & -\partial_{X_{t1}} & \partial_{X_{t3}} \\ 0 & 0 & \partial_{X_{t2}} & -\partial_{X_{t1}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0$$

Matrix Representation (continued):

$$\text{Moisil} - \text{Theodorescu}: \begin{bmatrix} 0 & -\partial_{x_{t1}} & -\partial_{x_{t2}} & -\partial_{x_{t3}} \\ \partial_{x_{t1}} & 0 & -\partial_{x_{t3}} & \partial_{x_{t2}} \\ \partial_{x_{t2}} & \partial_{x_{t3}} & 0 & -\partial_{x_{t1}} \\ \partial_{x_{t3}} & -\partial_{x_{t2}} & \partial_{x_{t1}} & 0 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0,$$

$$\text{Rarita} - \text{Scwhinger}: \begin{bmatrix} \partial_{X_{t1}} & \partial_{X_{t3}} & 0 & 0 \\ \partial_{X_{t2}} & \partial_{X_{t1}} & 2\partial_{X_{t3}} & 0 \\ 0 & 2\partial_{X_{t2}} & -\partial_{X_{t1}} & \partial_{X_{t3}} \\ 0 & 0 & \partial_{X_{t2}} & -\partial_{X_{t1}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0$$

FREE RESOLUTION

$n = 1, 2$

$$n = 1: \qquad 0 \longrightarrow R(-1)^4 \xrightarrow{P^T} R^4 \longrightarrow 0$$

$$n = 2: \qquad 0 \longrightarrow R(-4)^4 \xrightarrow{P_2^T} R(-3)^8 \xrightarrow{P_1^T} R(-1)^8 \xrightarrow{P^T} R^4 \longrightarrow 0$$

where P_1 and P_2 involves *Rarita–Schwinger like* operators (invariant?)

FREE RESOLUTION

$n \geq 3$

- Cauchy-Fueter and Rarita-Schwinger have *the same* free resolution:

$$0 \longrightarrow R^{\beta_{2n-1}}_{(-2n)} \longrightarrow \dots \longrightarrow R^{\beta_3}_{(-4)} \longrightarrow R^{\beta_2}_{(-3)} \longrightarrow R^{4n}_{(-1)} \xrightarrow{P^T} R^4 \longrightarrow 0$$

$$\beta_d = 4n \binom{2n-1}{d} \frac{d-1}{d+1}, \quad d > 1.$$

- It can be proved used standard Gröbner Basis techniques

Remarks

- Key ingredient: **Hilbert series** $\mathcal{H}_{R^4/M}(z) = 4\frac{1+(n-1)z}{(1-z)^{n+1}}$

- The algebraic dual of the free resolution is **exact**.

- This implies exactness at the level of sheaves [e.g. $\mathcal{S} = \mathcal{C}^\infty(\mathbb{R}^{3n})$]

$$0 \longrightarrow \mathcal{S}^P \longrightarrow \mathcal{S}^{\beta_0} \xrightarrow{P(D)} \mathcal{S}^{\beta_1} \xrightarrow{P_1(D)} \dots \longrightarrow \mathcal{S}^{\beta_{2n-1}} \longrightarrow 0$$

- Is it the same as the complex of *invariant operators*?

Compatibility conditions

- Key ingredient:

$$C_{ij} := \widetilde{RS}_i RS_j - \widetilde{RS}_j RS_i = 5(\partial_{X_{i2}}\partial_{X_{j3}} - \partial_{X_{i3}}\partial_{X_{j2}})I_4$$

- Relations at the first step **include**:

$$\boxed{(C_{ij} + RS_i\widetilde{RS}_j)g_i - RS_i\widetilde{RS}_i g_j = 0} \quad i \neq j$$

- Relations at the second step **include**:

$$\boxed{RS'{}_i h_{ij} - RS'_j h_{ji} = 0} \quad i < j$$

- This implies the existence of "exceptional" syzygies for $n > 2$...

Dimension Four

$m = 4$

Penrose notation: $\boxed{\nabla^{A\dot{B}}\varphi_{AB\dot{A}} = 0}$ Rarita-Scwhinger equation

$$\begin{bmatrix} \nabla^{0\dot{0}} & 0 & \nabla^{1\dot{0}} & 0 & 0 & 0 \\ 0 & \nabla^{0\dot{0}} & 0 & \nabla^{1\dot{0}} & 0 & 0 \\ 0 & 0 & \nabla^{0\dot{0}} & 0 & \nabla^{1\dot{0}} & 0 \\ 0 & \nabla^{0\dot{1}} & 0 & \nabla^{1\dot{1}} & 0 & 0 \\ 0 & 0 & \nabla^{0\dot{1}} & 0 & \nabla^{1\dot{1}} & 0 \\ 0 & 0 & 0 & \nabla^{0\dot{1}} & 0 & \nabla^{1\dot{1}} \end{bmatrix} \begin{bmatrix} \varphi_{00\dot{0}} \\ \varphi_{00\dot{1}} \\ \varphi_{10\dot{0}} \\ \varphi_{10\dot{1}} \\ \varphi_{11\dot{0}} \\ \varphi_{11\dot{1}} \end{bmatrix} = 0$$

Some Experiments

$m = 4$

- $n = 1 \quad : \; 0 \longrightarrow R^6 \xrightarrow{RS} R^6_{(1)} \longrightarrow 0$

- $n = 2 \quad : \; 0 \longrightarrow R^6 \xrightarrow{RS} R^{12}_{(1)} \xrightarrow{P_1} R^2_{(2)} \oplus R^8_{(3)} \longrightarrow R^4_{(4)} \longrightarrow 0$

- $n = 3 \quad :$

$$0 \to R^6 \xrightarrow{RS} R^{18}_{(1)} \xrightarrow{P_1} R^6_{(2)} \oplus R^{38}_{(3)} \longrightarrow R^{60}_{(4)} \longrightarrow R^{36}_{(5)} \longrightarrow R^8_{(6)} \to 0$$

References (this means it's over!)

• F. Colombo, I. Sabadini, F. Sommen, D. C. Struppa, *Analysis of Dirac systems and computational algebra*, Progress in Mathematical Physics, Vol. 39, *Birkhäuser*, Boston, 2004.

• CoCoA, A software package for COmputations in COmmutative Algebra, freely available at `http://cocoa.dima.unige.it`

• I. Sabadini, M. Shapiro, D.C. Struppa, *Algebraic analysis of the Moisil-Theodorescu system*, Compl. Var. **40** (2000), 333-357.