27^{th} Winter School GEOMETRY AND PHYSICS

Srní, 13-20 January 2007

Algebraic Resolutions for the Dirac Operator in Three Vector Variables

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OUTLINE

1. A non-commutative problem of syzygies

2. First tentative: Gröbner bases

3. Second tentative: megaforms

4. Last tentative: BGG graph

CLIFFORD ANALYSIS

Clifford algebra : $\mathcal{C}_n = \langle e_1, \ldots e_n \mid e_i e_j + e_j e_i = -2\delta_{ij},\ i, j = 1 \ldots n \rangle$

real variables $\underline{x}_1, \ldots, \underline{x}_k \in \mathbb{R}^n$, $\underline{x}_i = (x_{i1}, \ldots, x_{in}),\ i = 1 \ldots k$

differentiable functions $f : (\mathbb{R}^n)^k \longrightarrow \mathcal{C}_n$

Dirac operator $\partial_{\underline{x}_i} = \sum_j e_j \frac{\partial}{\partial x_{ij}},\ i = 1 \ldots k$

Dirac System defining **monogenic** functions $\left\{ \begin{array}{rcl} \partial_{\underline{x}_1} f & = & 0 \\ \vdots & & \\ \partial_{\underline{x}_k} f & = & 0 \end{array} \right.$

CLIFFORD ANALYSIS

$k = 3$

Clifford algebra : $\mathcal{C}_n = \langle e_1, \dots e_n \mid e_i e_j + e_j e_i = -2\delta_{ij},\ i,j = 1 \dots n \rangle$

real variables $\underline{x}_1, \underline{x}_2, \underline{x}_3 \in \mathbb{R}^n$, $\underline{x}_i = (x_{i1}, \dots, x_{in}),\ i = 1 \dots 3$

differentiable functions $f, \mathbf{g_1}, \mathbf{g_2}, \mathbf{g_3} : (\mathbb{R}^n)^3 \longrightarrow \mathcal{C}_n$

Dirac operator $\partial_{\underline{x}_i} = \sum_j e_j \frac{\partial}{\partial x_{ij}},\ i = 1 \dots 3$

Non-homogenous **Dirac System**

$$\begin{cases} \partial_{\underline{x}_1} f = \mathbf{g_1} \\ \partial_{\underline{x}_2} f = \mathbf{g_2} \\ \partial_{\underline{x}_3} f = \mathbf{g_3} \end{cases}$$

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Dirac operator $\partial_{\underline{x}_i} = \sum_j e_j \frac{\partial}{\partial x_{ij}},\ i = 1 \dots 3$

$$\begin{cases} \partial_{\underline{x}_1} f & = & \mathbf{g_1} \\ \partial_{\underline{x}_2} f & = & \mathbf{g_2} \\ \partial_{\underline{x}_3} f & = & \mathbf{g_3} \end{cases}$$

$$\boxed{P \cdot \mathbf{g_1} + Q \cdot \mathbf{g_2} + R \cdot \mathbf{g_3} = 0}$$

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$$\boxed{P \cdot \partial_{\underline{x}_1} + Q \cdot \partial_{\underline{x}_2} + R \cdot \partial_{\underline{x}_3} = 0}$$

NON COMMUTATIVE SYZYGIES

- Non Comm. Algebra : $\mathcal{C}_n \otimes \mathbb{C}[\partial_{x_{ij}}] = \dfrac{\langle e_1, \dots e_n, \partial_{x_{11}}, \dots, \partial_{x_{3n}} \rangle}{I}$

(a) $e_i e_j = -e_j e_i$, for all $i \neq j$

(b) $e_i \partial_x = \partial_x e_i$, for all i, x

(c) $\partial_x \partial_y = \partial_y \partial_x$, for all x, y

(d) $\boxed{e_i^2 = -1}$, for all i

- (a),(b),(c) = **solvable algebra** ⇒ Gröbner basis theory ⇒ syzygies.

- Also (d) = there are **idempotents** ... problems of minimality!

ALGEBRAIC ANALYSIS

$n \geq 6$

- compact form of the system: $P(D)f = g$

- associate module R^{2^n}/M, where $M = \langle$ rows of $P \rangle$,
$R = \mathbb{C}[\,\partial_{x_{ij}} \mid i = 1 \ldots n, j = 1 \ldots 3]$

- we can construct the **free resolution** for R^{2^n}/M:

$$0 \to R^{2^n} \longrightarrow R^{\mathbf{3}\cdot 2^n} \to R^{\mathbf{8}\cdot 2^n} \to R^{\mathbf{12}\cdot 2^n} \to R^{\mathbf{8}\cdot 2^n} \to R^{\mathbf{3}\cdot 2^n} \xrightarrow{P^T} R^{2^n} \to 0$$

- Its dual is a **complex** of morphisms of free R-modules:

$$0 \to R^{2^n} \xrightarrow{P} R^{\mathbf{3}\cdot 2^n} \to R^{\mathbf{8}\cdot 2^n} \to R^{\mathbf{12}\cdot 2^n} \to R^{\mathbf{8}\cdot 2^n} \to R^{\mathbf{3}\cdot 2^n} \longrightarrow R^{2^n} \to 0$$

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$$0 \to R \longrightarrow R^{\mathbf{3}} \to R^{\mathbf{8}} \to R^{\mathbf{12}} \to R^{\mathbf{8}} \to R^{\mathbf{3}} \xrightarrow{P^T} R \to 0$$

- Its dual is a **complex** of morphisms of free R-modules:

$$0 \to R \xrightarrow{P} R^{\mathbf{3}} \xrightarrow{S} R^{\mathbf{8}} \to R^{\mathbf{12}} \to R^{\mathbf{8}} \to R^{\mathbf{3}} \longrightarrow R \to 0$$

ALGEBRAIC ANALYSIS

$\boxed{S \in \mathrm{Mat}_{192 \times 512}(R)}$!!

- Betti numbers with CoCoA then take a GUESS...

$$
\begin{cases}
\partial_{\underline{x}_2}\partial_{\underline{x}_1} g_1 - \partial^2_{\underline{x}_1} g_2 = 0 \\
\partial_{\underline{x}_3}\partial_{\underline{x}_1} g_1 - \partial^2_{\underline{x}_1} g_3 = 0 \\
\partial_{\underline{x}_1}\partial_{\underline{x}_2} g_2 - \partial^2_{\underline{x}_2} g_1 = 0 \\
\partial_{\underline{x}_3}\partial_{\underline{x}_2} g_2 - \partial^2_{\underline{x}_2} g_3 = 0 \\
\partial_{\underline{x}_1}\partial_{\underline{x}_3} g_3 - \partial^2_{\underline{x}_3} g_1 = 0 \\
\partial_{\underline{x}_2}\partial_{\underline{x}_3} g_3 - \partial^2_{\underline{x}_3} g_2 = 0 \\
\{\partial_{\underline{x}_2}, \partial_{\underline{x}_3}\} g_1 - \partial_{\underline{x}_1}\partial_{\underline{x}_2} g_3 - \partial_{\underline{x}_1}\partial_{\underline{x}_3} g_2 = 0 \\
\{\partial_{\underline{x}_1}, \partial_{\underline{x}_3}\} g_2 - \partial_{\underline{x}_2}\partial_{\underline{x}_1} g_3 - \partial_{\underline{x}_2}\partial_{\underline{x}_3} g_1 = 0
\end{cases}
$$

$$
0 \to R \xrightarrow{P} R^{\mathbf{3}} \xrightarrow{S} R^{\mathbf{8}} \to R^{\mathbf{12}} \to R^{\mathbf{8}} \to R^{\mathbf{3}} \longrightarrow R \to 0
$$

The Dirac derivatives satisfy special cubic non commutative relations:

$$\boxed{r_{ij\ell} := [\{\partial_i, \partial_j\}, \partial_\ell] = 0}$$

- **Radial algebra**: associative algebra $\mathcal{R} = \langle \partial_1, \ldots, \partial_k \rangle$ satisfying the radial relations $r_{ij\ell} = 0$, e.g.

$$\partial_1 \partial_2 \partial_3 = \partial_3 \partial_2 \partial_1 + \partial_3 \partial_1 \partial_2 - \partial_2 \partial_1 \partial_3 \quad \text{and} \qquad \partial_i^2 \partial_j = \partial_j \partial_i^2$$

- There does not exist a *proper* theory of Gröbner bases for the radial algebra, but there is a **normal form** for monomials in $\mathcal{R}$

$$\boxed{\partial_{i_1} \cdots \partial_{i_s} \cdot \partial_{j_1}^2 \cdots \partial_{j_t}^2} \qquad i_a \neq \min(i_a, i_{a+1}, i_{a+2}), \quad j_1 \leq \ldots \leq j_t$$

MEGAFORMS

linear megaforms $D^i_1, D^i_2, D^i_3, \quad i \in \mathbb{N}$

quadratic megaforms: $D^i_{11}, D^i_{12}, D^i_{21}, D^i_{13} \dots \quad i \in \mathbb{N}$

$$F_0 = C^\infty((\mathbb{R}^n)^3, \mathcal{C}_n)$$

$$d^0 = D^0_1\partial_1 + D^0_2\partial_2 + D^0_3\partial_3 \ : F_0 \longrightarrow F_1$$

F_1: $\mathbf{g} = D^0_1 g_1 + D^0_2 g_2 + D^0_3 g_3, \quad g_i \in F_0$

$$d^1 = D^1_1\partial_1 + D^1_2\partial_2 + D^1_3\partial_3 + D^1_{11}\partial_1\partial_1 + D^1_{12}\partial_1\partial_2 + D^1_{21}\partial_2\partial_1 + \cdots + \cdots$$

F_2 : $\mathbf{h} = D^1_1 h_1 + D^1_2 h_2 + D^1_3 h_3 \ + \ D^1_{11} h_{11} + D^1_{12} h_{12} + \cdots + D^1_{33} h_{33}$

ALGORITHM

- *reduce* $d^1 d^0 = 0$ using $r_{ij\ell} = 0$

$\longrightarrow$ find **relations on megaforms** (analogous to $dx \wedge dy = -dy \wedge dx$)

- *reduce* $d^1\mathbf{g} = 0$ using the relations on megaforms

$\longrightarrow$ find the **first syzygies**, define $\mathbf{h} = d^1\mathbf{g}$

- *reduce* $d^2 d^1 = 0$ using $r_{ij\ell} = 0$ $\quad \ldots$

$$0 \longrightarrow F_0 \xrightarrow{d^0} F_1 \xrightarrow{d^1} F_2 \xrightarrow{d^2} \ldots \xrightarrow{d^{N-2}} F_{N-1} \xrightarrow{d^{N-1}} F_N \longrightarrow \ldots$$

EXAMPLE k=2

$$d^1 d^0 = D_{11}D_1\partial_1^3 + D_{12}D_1\partial_1\partial_2\partial_1 + D_{21}D_1\partial_2\partial_1^2 + D_{22}D_1\partial_2^2\partial_1 +$$
$$+ D_{11}D_2\partial_1^2\partial_2 + D_{12}D_2\partial_1\partial_2^2 + D_{21}D_2\partial_2\partial_1\partial_2 + D_{22}D_2\partial_2^3 = 0$$

$$\downarrow$$

$$(D_{21}D_1 + D_{11}D_2)\partial_2\partial_1^2 + (D_{22}D_1 + D_{12}D_2)\partial_1\partial_2^2 +$$
$$+ D_{11}D_1\partial_1^3 + D_{12}D_1\partial_1\partial_2\partial_1 + D_{21}D_2\partial_2\partial_1\partial_2 + D_{22}D_2\partial_2^3 = 0$$

$$\downarrow$$

$$D_{11}D_1 = D_{12}D_1 = D_{21}D_2 = D_{22}D_2 = 0$$
$$D_{11}D_2 + D_{21}D_1 = D_{22}D_1 + D_{12}D_2 = 0$$

EXAMPLE k=2 (cont)

$$\text{using} \begin{cases} D_{11}D_1 = D_{12}D_1 = D_{21}D_2 = D_{22}D_2 = 0 \\ D_{11}D_2 + D_{21}D_1 = D_{22}D_1 + D_{12}D_2 = 0 \end{cases}$$

$$d^1\mathbf{g} = D_{11}D_1\partial_1^2 g_1 + D_{11}D_2\partial_1^2 g_2 + D_{12}D_1\partial_1\partial_2 g_1 + D_{12}D_2\partial_1\partial_2 g_2$$

$$+D_{21}D_1\partial_2\partial_1 g_1 + D_{21}D_2\partial_2\partial_1 g_2 + D_{22}D_1\partial_2^2 g_1 + D_{22}D_2\partial_2^2 g_2 = 0.$$

$$\downarrow$$

$$D_{21}D_1(\partial_1^2 g_2 - \partial_2\partial_1 g_1) + D_{12}D_2(\partial_2^2 g_1 - \partial_1\partial_2 g_2) = 0$$

compatibility conditions: $\partial_1^2 g_2 - \partial_2\partial_1 g_1 = 0$, $\partial_2^2 g_1 - \partial_1\partial_2 g_2$

GEOMETRIC APPROACH

We can study **invariance** w.r.t. the reductive part

$P :=$ SL$(k)\otimes$Spin(n) of the Lie group $G =$Spin$(n+k,k)$

- Spinor representation $\mathbb{S}$ of Spin(n)
- Trivial and standard representation $\mathbb{C}$ and $\mathbb{C}^k$ of SL(k)

FACT: There exists (up to multiples) a unique G-invariant diff. op.

$$\Gamma(G \times_P (\mathbb{C} \otimes \mathbb{S})) \xrightarrow{D_k} \Gamma(G \times_P (\mathbb{C}^k \otimes \mathbb{S}))$$

which in suitable coordinates corresponds to the Dirac operator in several variables

• Operators are dual to Verma module morphisms $M_{\mathfrak{p}}(\mathbb{V}) \to M_{\mathfrak{p}}(\mathbb{W})$

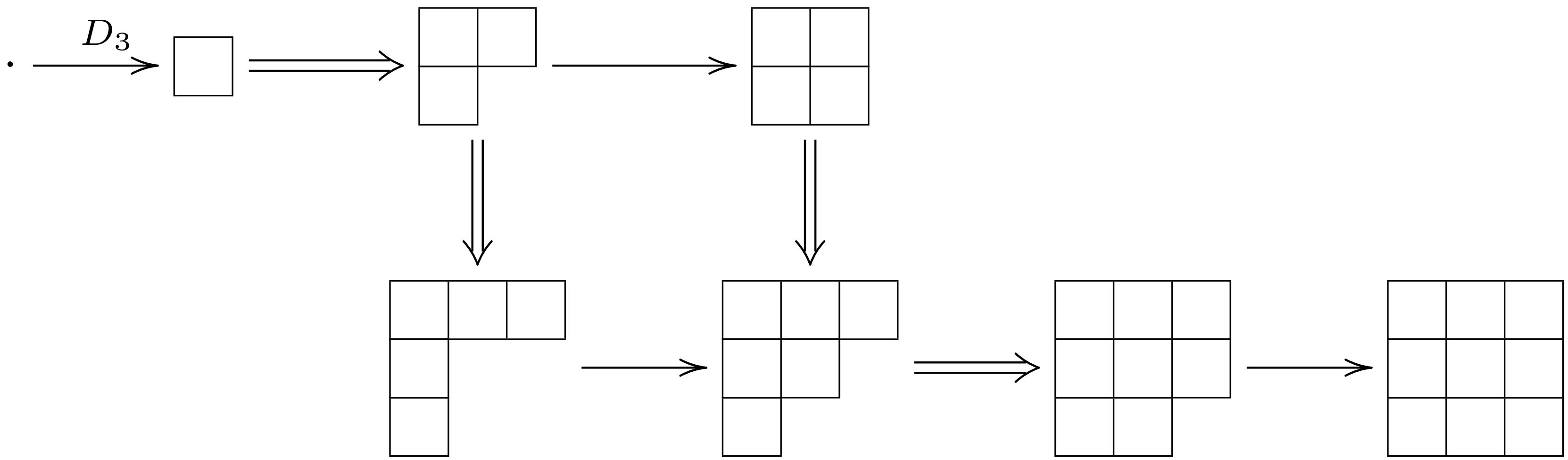
• The affine orbit of the Weyl group of $\mathfrak{g}$ gives a sequence of morphisms forming the **BGG graph**.

IF n odd and such that $n \geq 2k$ **THEN** it is possible to **describe explicitly** the sequence of operators recursively, using only representation spaces of the type $\mathbb{V} \otimes \mathbb{S}$ where $\mathbb{V}$ is a $\mathrm{SL}(k)$ module.

A conjecture is given for n even and it is similar.

BGG SEQUENCE

n odd, $k = 3$



$$0 \to R \xrightarrow{P} R^{\mathbf{3}} \xrightarrow{S} R^{\mathbf{8}} \to R^{\mathbf{12}} \to R^{\mathbf{8}} \to R^{\mathbf{3}} \longrightarrow R \to 0$$

BGG SEQUENCE

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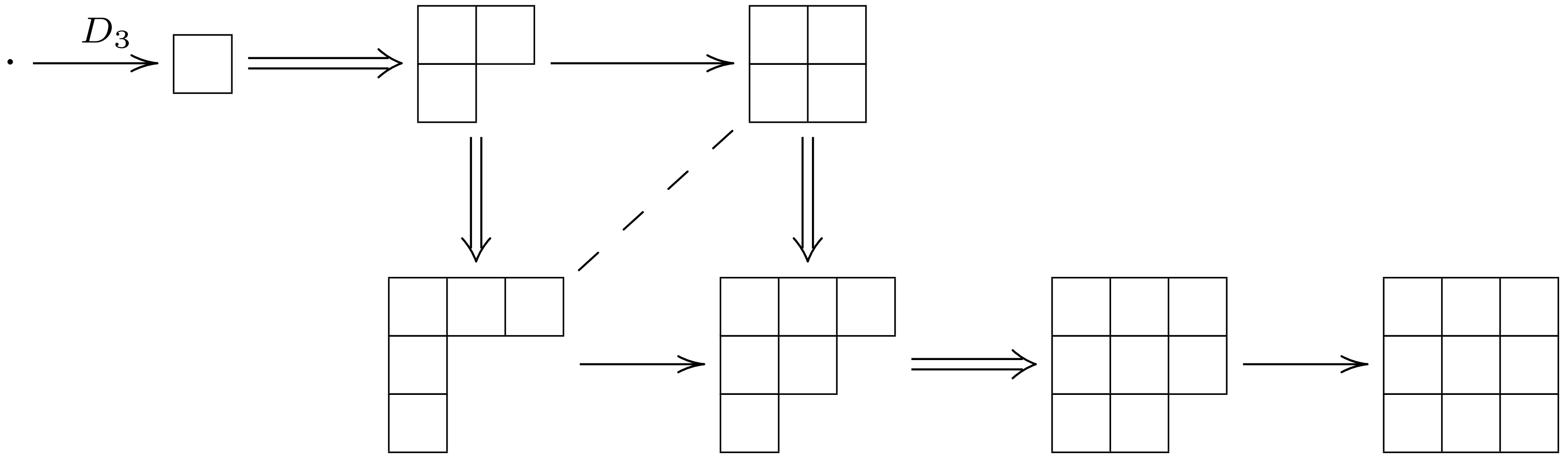

$$0 \to R \xrightarrow{P} R^{\mathbf{3}} \xrightarrow{S} R^{\mathbf{8}} \to \begin{matrix} R^{\mathbf{6}} \\ \oplus \\ R^{\mathbf{6}} \end{matrix} \to R^{\mathbf{8}} \to R^{\mathbf{3}} \longrightarrow R \to 0$$

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