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The Dolbeault complex for several Clifford variables

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OUTLINE

1. Several Dirac operators in **representation theory**

2. Several Dirac operators in **algebraic analysis**

3. Where do the two approaches **come together**?

Clifford algebra : $\mathcal{C}_n = \langle e_1, \dots e_n \mid e_i e_j + e_j e_i = -2\delta_{ij},\ i,j = 1 \dots n \rangle$

real variables $\underline{x}_1, \dots, \underline{x}_k \in \mathbb{R}^n, \quad \underline{x}_i = (x_{i1}, \dots, x_{in}),\ i = 1 \dots k$

differentiable functions $f : (\mathbb{R}^n)^k \longrightarrow \mathcal{C}_n$

Dirac operator $\partial_{\underline{x}_i} = \sum_j e_j \frac{\partial}{\partial x_{ij}},\ i = 1 \dots k$

Dirac System defining **monogenic** functions

$$\begin{cases} \partial_{\underline{x}_1} f & = & 0 \\ \vdots & & \\ \partial_{\underline{x}_k} f & = & 0 \end{cases}$$

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Non-homogenous **Dirac System**

$$\begin{cases} \partial_{\underline{x}_1} f &= \mathbf{g_1} \\ \vdots & \\ \partial_{\underline{x}_k} f &= \mathbf{g_k} \end{cases}$$

GEOMETRIC APPROACH

P. Franek

We can study **invariance** w.r.t. the reductive part

$P :=$ SL$(k)\otimes$Spin(n) of the Lie group $G =$Spin$(n+k,k)$

- Spinor representation $\mathbb{S}$ of Spin(n)
- Trivial and standard representation $\mathbb{C}$ and $\mathbb{C}^k$ of SL(k)

FACT: There exists (up to multiples) a unique G-invariant diff. op.

$$\Gamma(G \times_P (\mathbb{C} \otimes \mathbb{S})) \xrightarrow{D_k} \Gamma(G \times_P (\mathbb{C}^k \otimes \mathbb{S}))$$

which in suitable coordinates corresponds to the Dirac operator in several variables

• Operators are dual to Verma module morphisms $M_{\mathfrak{p}}(\mathbb{V}) \to M_{\mathfrak{p}}(\mathbb{W})$

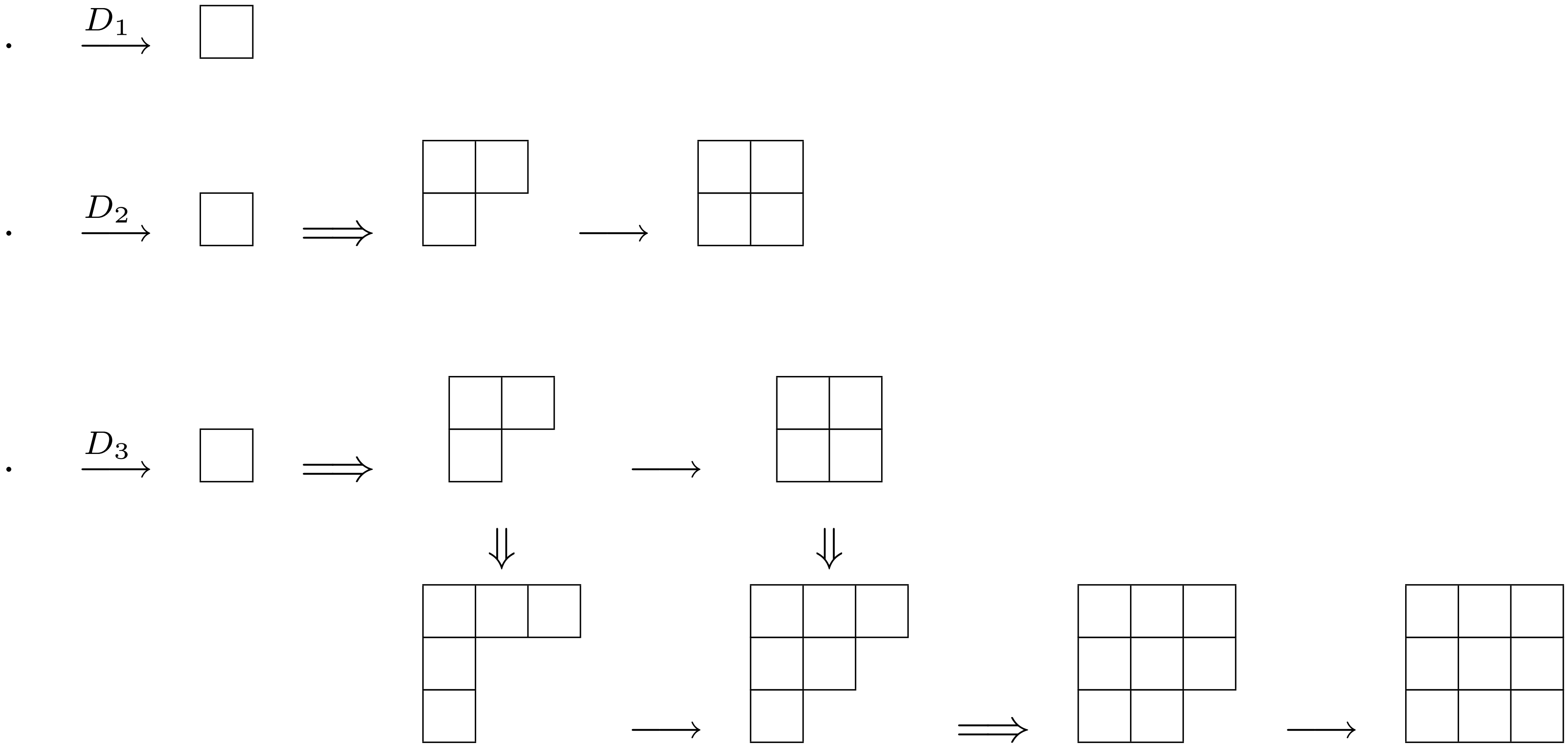
• The affine orbit of the Weyl group of $\mathfrak{g}$ gives a sequence of morphisms forming the **BGG graph**.

IF n odd and such that $n \geq 2k$ **THEN** it is possible to **describe explicitly** the sequence of operators recursively, using only representation spaces of the type $\mathbb{V} \otimes \mathbb{S}$ where $\mathbb{V}$ is a $\mathrm{SL}(k)$ module.

A conjecture is given for n even and it is similar.

BGG SEQUENCE

n odd, $k = 1, 2, 3$



ALGEBRAIC ANALYSIS

$n \geq 2k$

- compact form of the system: $P_{kn}(D)f = g$

- associate module R^{2^n}/M_{kn}, where $M_{kn} = \langle$ rows of $P_{kn} \rangle$,
$R = \mathbb{C}[\,\partial x_{ij} \mid i = 1 \ldots n, j = 1 \ldots k]$

- we are interested in the **free resolution** (exact complex):

$$0 \longrightarrow R^{\beta_s} \longrightarrow R^{\beta_{s-1}} \longrightarrow \ldots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow R^4/M \longrightarrow 0$$

- Its dual is a **complex** of morphisms of free R-module:

$$0 \longrightarrow R^{\beta_0} \longrightarrow R^{\beta_1} \longrightarrow \ldots \longrightarrow R^{\beta_{s-1}} \longrightarrow R^{\beta_s} \longrightarrow 0$$

DIMENSIONS

The complex is actually graded $0 \to \dots \longrightarrow \oplus_{j\in\mathbb{N}} R(-j)^{\beta_{ij}} \longrightarrow \cdots \to 0$
$\beta_{ij} \neq 0$ only for finite i, j

Hilbert Series : $H_{kn}(z) = 2^n \dfrac{\sum_{i,j}(-1)^i \beta_{ij} z^j}{(1-z)^{kn}} = 2^n \dfrac{(1+z)^{\binom{k}{2}}(1-z)^{\binom{k+1}{2}}}{(1-z)^{kn}}$

$\mathbf{B}_k^{\mathbf{L}} := \sum_{i=1}^{s}(-1)^i \beta_{iL} = \frac{1}{L!}\frac{d^L}{dz^L}|_{z=0}\,(1+z)^{\binom{k}{2}}(1-z)^{\binom{k+1}{2}}$

Weyl dimension: $\mathbf{d}_k^{\lambda} = \frac{(\lambda+\rho)!}{\rho!\,\Pi h}$, $\lambda = (\lambda_1, \dots, \lambda_k)$, $\rho = (k-1, \dots, 2, 1, 0)$

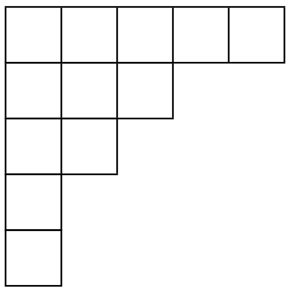
CONJECTURE

$n \geq 2k$

Conjecture 1. The dimensions of the spaces in the BGG and the Betti numbers are compatible:

$$\mathbf{B}_k^{\mathbf{L}} = \sum_{\lambda=\lambda' \, |\lambda|=\mathbf{L}} (-1)^{\sigma(\lambda)} \mathbf{d}_k^{\lambda}, \quad \forall \, k, \mathbf{L}$$

where $\sigma(\lambda) = \frac{\mathbf{L}+d}{2}$ is the number of boxes **on and below the diagonal** of a self-conjugate Young diagram λ with L boxes

$\sigma($  $) = 7$ is the **distance** from the first space in the BGG

ATTEMPTS

$$\mathbf{B}_k^{\mathbf{L}} = \sum_{j=\max(0,\mathbf{L}-\binom{k}{2})}^{\min(\mathbf{L},\binom{k+1}{2})} (-1)^j \binom{\binom{k}{2}}{j} \binom{\binom{k+1}{2}}{\mathbf{L}-j}$$

$$\mathbf{d}_{k+1}^{(\lambda_1,\dots,\lambda_k,0)} = c(\lambda,k)\mathbf{d}_k^{(\lambda_1,\dots,\lambda_k)}$$

- We tried recursive formulas, hypergeometric expansions... ??

- In particular it is hard to use induction on L for the RHS of C1

- Experimental evidence for $1 \leq k \leq 6$ and all relevant $\mathbf{L}$

CONJECTURE $(n \geq 2k)$

Conjecture 2. The geometric sequence constructed is a complex.

Motivation: The algebraic sequence is a complex by definition!

- If we also prove that they are exact, than they coincide because of the uniqueness of the minimal free resolution

- **IDEA** Use the **radial relations** satisfied by the Dirac operators

- I. Sabadini, F. Sommen, D.C. Struppa, *The Dirac complex on abstract vector variables: megaforms*, Exp. Math., **12** 2003, 351–364.

IDEA

Special cubic non commutative polynomials: $r_{ij\ell} := [\{\partial_i, \partial_j\}, \partial_\ell]$.

Radial algebra: associative algebra $\mathcal{R} = \langle \partial_1, \dots, \partial_k \rangle$ satisfying the radial relations $r_{ij\ell} = 0$, e.g.

$$\partial_1 \partial_2 \partial_3 = \partial_3 \partial_2 \partial_1 + \partial_3 \partial_1 \partial_2 - \partial_2 \partial_1 \partial_3 \quad \text{and} \qquad \partial_i^2 \partial_j = \partial_j \partial_i^2$$

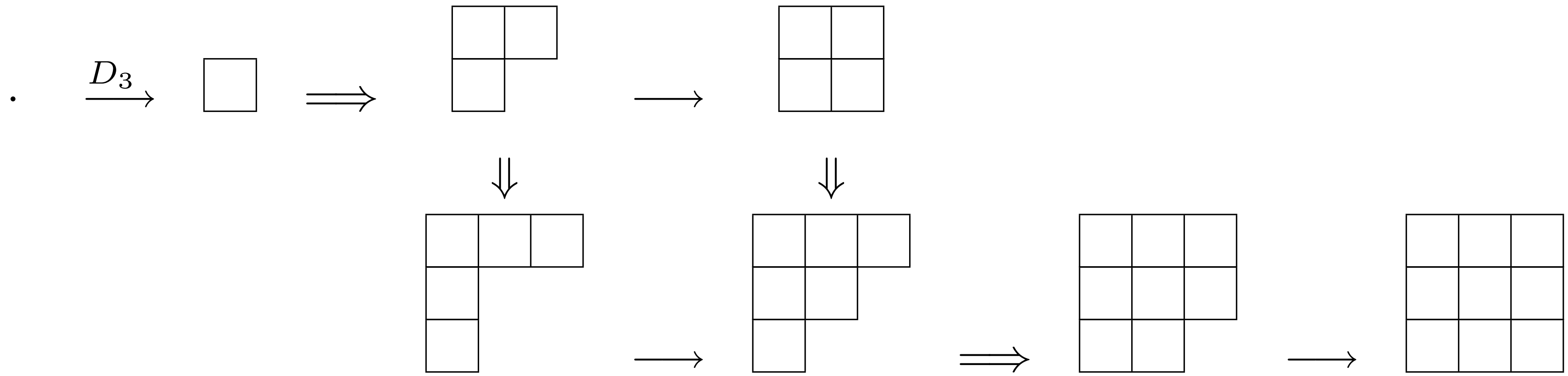
• There does not exist a *proper* theory of Gröbner bases for the radial algebra, but there is a **normal form** for monomials in $\mathcal{R}$

$$\boxed{\partial_{i_1} \cdots \partial_{i_s} \cdot \partial_{j_1}^2 \cdots \partial_{j_t}^2}$$

$$i_a \neq \min(i_a, i_{a+1}, i_{a+2}), \; a = 1 \dots s-2, \quad \text{and} \quad j_1 \leq \dots \leq j_t$$

ATTEMPTS

an algorithmic approach



• write a projection of $g_{bc\ldots e} = \partial_b g_{c\ldots e}$ or $g_{abc\ldots e} = \partial_a \partial_b g_{c\ldots e}$ onto one of the spaces $\mathbb{W}_\lambda \otimes \mathbb{S}$ where $g_{c\ldots e}$ comes from a previous space in the graph.

• use radial relations to rewrite $\sum \partial_a \partial_b \partial_c g_{d\ldots e}$

References

- F. Colombo, I. Sabadini, F. Sommen, D. C. Struppa, *Analysis of Dirac systems and computational algebra*, Progress in Mathematical Physics, *Birkhäuser*, Boston, 2004.

- P. Franek, *Several Dirac Operators in Parabolic Geometry*, Ph.D. Dissertation, Charles University, Prague 2006.

- I. Sabadini, F. Sommen, D.C. Struppa, *The Dirac complex on abstract vector variables: megaforms*, Exp. Math., **12** 2003, 351–364.