Algebra and Combinatorics Seminar

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Computation of Noetherian Operators

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Outline

1) **Motivations:** three apparently different questions to answer

2) **Definitions**: multiplicity variety and noetherian operators

3) **Examples**:

- Euler's formula for ODEs
- the principal case (wave equation)
- the *zerodimensional* case...

4) **The Algorithm** to compute noetherian operators

5) **Demo with** CoCoA

Question 1

Given an ideal I in $R := \mathbb{C}[x_1, \ldots, x_n]$, how do we *test* if $\mathbf{f} \in \mathbf{I}$ without using a division algorithm?

Example

$I := (\mathbf{x^2}, \mathbf{xy} - y) \subset \mathbb{C}[x, y]$, let us test if $f = xy - 2y \in I$ using the division algorithm:

$x^2y - 2y \xrightarrow{x^2} -2y$ cannot be reduced further

$x^2y - 2y \xrightarrow{xy-y} y^2 - 2y$ cannot be reduced either

- we could to use a Gröbner Basis for I: $(\mathbf{x^2}, \mathbf{y})$

$x^2y - 2y \xrightarrow{x^2} -2y \xrightarrow{y} 0$

I am looking for an alternative way to test whether $f \in I$

Question 2

How do we describe the algebraic set $V(I)$ so that we take into account *geometric multiplicities*?

Nullstellensatz:

$$\{\text{algebraic sets of } \mathbb{C}^n\} \leftrightarrow \{\text{radical ideals of } \mathbb{C}[x_1, \ldots, x_n]\}$$

Example

$I := (y^2, x^2 - y) \subset \mathbb{C}[x, y]$, this represents the origin with multiplicity 4

$V = V(I) = \{(0, 0)\}$

$I(V) = \sqrt{I} = (x, y)$ we have lost the multiplicity of the root

I am looking for a geometric object to characterize I taking into account that not only the generators vanish at the origin but also some of their *derivatives*

Question 3

How do we represent explicitly all the solutions to a system of linear constant coefficients *partial differential equations*?

Example 1:

$$\begin{cases} \dfrac{\partial f}{\partial x} & = & -g \\ \\ \dfrac{\partial f}{\partial y} & = & -\dfrac{\partial g}{\partial x} \\ \\ \dfrac{\partial g}{\partial y} & = & 0 \end{cases}$$

$\rightarrow$ in this case it could be done "by hand":

$f(x, y) = \frac{1}{2}ax^2 - bx + ay + c$

$g(x, y) = -ax + b$

Example 2: Regular functions $f : \mathbb{H}^n \longrightarrow \mathbb{H}$

$f = f_0 + if_1 + jf_2 + kf_3$, $q_i = x_{i0} + ix_{i1} + jx_{i2} + kx_{i3}$

$$\frac{\partial f(q_i)}{\partial \bar{q_i}} = 0 \Leftrightarrow \begin{cases} \frac{\partial f_0}{\partial x_{i0}} - \frac{\partial f_1}{\partial x_{i1}} - \frac{\partial f_2}{\partial x_{i2}} - \frac{\partial f_3}{\partial x_{i3}} & = 0 \\ \frac{\partial f_0}{\partial x_{i1}} + \frac{\partial f_1}{\partial x_{i0}} - \frac{\partial f_2}{\partial x_{i3}} + \frac{\partial f_3}{\partial x_{i2}} & = 0 \\ \frac{\partial f_0}{\partial x_{i2}} + \frac{\partial f_1}{\partial x_{i3}} + \frac{\partial f_2}{\partial x_{i0}} - \frac{\partial f_3}{\partial x_{i1}} & = 0 \\ \frac{\partial f_0}{\partial x_{i3}} - \frac{\partial f_1}{\partial x_{i2}} + \frac{\partial f_2}{\partial x_{i1}} + \frac{\partial f_3}{\partial x_{i0}} & = 0 \end{cases}$$

f is *regular* if $\frac{\partial f(q_1)}{\partial \bar{q_1}} = \cdots = \frac{\partial f(q_n)}{\partial \bar{q_n}} = 0$

Even if it looks like a different question, we can transform it into a more algebraic problem: we can replace each derivative with a commutative variable:

$$x \leftrightarrow \frac{\partial}{\partial x}, \quad , y \leftrightarrow \frac{\partial}{\partial y}, \dots \quad p(x,y) \leftrightarrow p(D) = p(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$$

and then look at a system of PDEs as an ideal or module:

$$\left\{ \begin{array}{rcl} \frac{\partial^2}{\partial y^2} f & = & 0 \\ & & \\ \frac{\partial^2}{\partial x^2} f - \frac{\partial}{\partial y} f & = & 0 \end{array} \right. \rightarrow I = (y^2, x^2 - y)$$

$$\begin{cases} \dfrac{\partial f}{\partial x} + g &= 0 \\ \dfrac{\partial f}{\partial y} + \dfrac{\partial g}{\partial x} &= 0 \\ \dfrac{\partial g}{\partial y} &= 0 \end{cases} \rightarrow M = \langle (x,1), (y,x), (0,y) \rangle$$

Finding solutions to a system of equations is *equivalent* to the membership problem for the associated ideal or module!

Answer to Question i, $i = 1, 2, 3$:

Multiplicity Variety

Let V_j be algebraic varieties in $\mathbb{C}^n$ and
let ∂_j be differential operators with polynomial coefficients, for
$j = 1, ..., t$. Then we call

$$\mathbf{V} = \{(V_1, \partial_1); (V_2, \partial_2); \ldots; (V_t, \partial_t)\}$$

a multiplicity variety.
The operators ∂_j's are called *noetherian operators*

Theorem Let I be an ideal of R. Then there exists a multiplicity variety V such that a polynomial f belongs to I if and only if $\partial_j f_{|V_j} = 0$ for every $j = 1, \ldots, t$.

A new ideal membership test

$I = (f, g) = (y^2, x^2 - y) \to V = \{(V_1, \partial_1), \ldots, (V_4, \partial_4)\}$ where

$V_i = \{(0,0)\}$

$\partial_1 = \underline{1},\ \partial_2 = \underline{\partial x},\ \partial_3 = \frac{1}{2}\underline{\partial x^2 + \partial y},\ \partial_4 =,\ \frac{1}{6}\underline{\partial x^3 + \partial x \partial y}$

Application: we can test if $h = x^2 y + y^2$ belongs to I:

$\partial_1 h_{|(0,0)} = x^2 y + y^2_{|(0,0)} = 0, \qquad \partial_2 h_{|(0,0)} = 2xy_{|(0,0)} = 0$

$\partial_3 h_{|(0,0)} = x^2 + 3y_{|(0,0)} = 0, \qquad \partial_4 h_{|(0,0)} = 0$

$\to$ in fact $h = 2f + yg$

Theorem (Fundamental Principle of Ehrenpreis - Palamodov, 1960) Let $p_1(D), \ldots, p_r(D)$ be linear constant coefficients partial differential operators in n variables. Then the multiplicity variety $\{(V_1, \partial_1); \ldots; (V_t, \partial_t)\}$ associated to $I = (p_1, \ldots, p_r)$ is such that every function $f \in \mathbb{C}^\infty(\mathbb{R}^n)$ satisfying

$$p_1(D)f = \ldots = p_r(D)f = 0$$

can be represented as

$$f(x) = \sum_{j=1}^{t} \int_{V_j} \partial_j(e^{ix\cdot z}) d\nu_j(z), \tag{1}$$

for suitable Radon measures $d\nu_j$.

The above (1) is an integral representation of the solutions of the system of equations associated to $I = (p_1, \ldots, p_r)$. It generalizes the well known **Euler's formula**

How do we actually compute a multiplicity variety?

$$V = \{(V_1, \partial_1); (V_2, \partial_2); \ldots; (V_t, \partial_t)\}$$

Primary Decomposition: $I = Q_1 \cap \cdots \cap Q_t$ gives

$$V(I) = V(Q_1) \cup \cdots \cup V(Q_t)$$

so we can define $V_j = V(Q_j)$ and then we just look for the ∂_j's to attach to each irreducible component.

Singular → `www.singular.uni-kl.de`
CoCoA → `cocoa.dima.unige.it`

How do we compute the noetherian operators ∂_j's?

Case 1 (Euler's formula)

If $n = 1$ we have $I = (q(x)^\alpha)$ where $q(x) = x - a$ is irreducible. The operators in this case are simply

$$id, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}, \ldots, \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}$$

Example
Given the ODE $\frac{\partial^\alpha f}{\partial z} = af$ its symbol is $q(x) = (x-a)^\alpha$ and using the integral formula (1) we get $f(z) =$

$$= c_0(e^{xz})_{|x=a} + \cdots + c_{\alpha-1}\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}(e^{xz})_{|x=a} =$$

$$= c_0e^{az} + c_1ze^{az} + c_2z^2e^{az} + \cdots + c_{\alpha-1}z^{\alpha-1}e^{az}$$

Case 2 (Principal Ideals)

If $n \geq 1$ we have $I = (q(x_1, \ldots, x_n)^\alpha)$, $q(x_1, \ldots, x_n)$ irreducible. Again

$$id, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial^{\alpha-1}}{\partial x_1^{\alpha-1}}$$

a long as x_1 appears as a simple power in q (*normal position*):

$$q = x_1^d + p_{d-1}(x_2, \ldots, x_n)x_1^{d-1} + \cdots + p_0(x_2, \ldots, x_n)$$

Example: Wave Equation

$$\frac{\partial^2}{\partial z^2}u(z,t) - \frac{\partial^2}{\partial t^2}u(z,t) = 0$$

$q(x,y) = x^2 - y^2 = (x+y)(x-y)$

$I = (q) = (x+y) \cap (x-y)$ and each component has multiplicity one

$\mathbf{V} = \{(x+y=0, id); (x-y=0, id)\}$

$u(z,t) = F(z+t) + G(z-t)$

Case 3 (Zerodimensional Ideals)

$$t = x_1^{i_1} \cdots x_n^{i_n}, \quad D(t) = D(i_1, \ldots, i_n) = \tfrac{1}{i_1! \cdots i_n!} \partial x_1^{i_1} \cdots \partial x_n^{i_n}$$

Space of differential operators:
$\mathcal{D} = \mathrm{Span}_{\mathbb{C}}(\{D(t) | t \,\text{is a term in}\, R\}) = \mathbb{C}[\partial x_1, \ldots, \partial x_n]$

Derivative-like morphisms σ_j:

$$\sigma_{x_j} D(i_1, \ldots, i_n) = \begin{cases} D(\ldots, i_j - 1, \ldots) & \text{if } i_j > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$D := D(0,2) = D(y^2) = \frac{1}{2} \frac{\partial^2}{\partial y^2} \to \sigma_y(D) = D(0,1) = \frac{\partial}{\partial y}$$

Closed Subspaces

Consider $\mathcal{D} = \mathbb{C}[\partial x_1, \ldots, \partial x_n]$ and extend σ_{x_j} to $\mathcal{D}$. A subspace U of $\mathcal{D}$ is *closed* if

$$\sigma_j(U) \subseteq U, \ \forall j$$

Example: $\langle 1, \quad \partial x, \quad \frac{1}{2}\partial x^2 + \partial y, \quad \frac{1}{6}\partial x^3 + \partial x \partial y \rangle$

This subspace is closed, because the subspace generated by noetherian operators is always closed...

Correspondence $\Delta \longleftrightarrow \mathcal{I}$

Consider a primary ideal I of R is primary and a subset U of $\mathcal{D}$. Suppose that $V(I) = \{(0, \ldots, 0)\}$.

$$\Delta(I) := \{L \in \mathcal{D} \,|\, L(f)(0, \ldots, 0) = 0 \;\forall f \in I\}.$$

$$\mathcal{I}(U) := \{f \in R \,|\, L(f)(0, \ldots, 0) = 0 \;\forall L \in U\}$$

Theorem Let $\mathfrak{m} = (x_1, \ldots, x_n)$. There is a **one to one** correspondence:

$$\{\mathfrak{m} - \text{primary ideals in } R\} \underset{\mathcal{I}}{\overset{\Delta}{\rightleftarrows}} \{\text{closed subspaces of } \mathcal{D}\}$$

The set $\Delta(I)$ is exactly the space generated by the noetherian operators associated to I!

Consequences:

Consider a primary zerodimensional ideal I centered at the origin with multiplicity μ:

- the noetherian operators associated to I are exactly $\mu = \dim_{\mathbb{C}}(\Delta(I))$

- the identity $id_{\mathcal{D}}$ is always a noetherian operator

- the maximum degree of an operator is $\mu - 1$

These facts and the use of Gröbner Bases techniques lead to an algorithm that can be easily implemented in CoCoA

Algorithm

input: a Gröbner Basis $\mathcal{G}$ of I **output**: $\Delta(I) = \{L_\beta\}$

1) Multiplicity of $\mu(I) = \dim_{\mathbb{C}}(R/I)$

2) Taylor expansion at the origin of a polynomial $h \in R$:

$$T_{\mu-1}h(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{N}^n,\, |\alpha| \leq \mu-1} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

3) Normal Form: $\mathrm{NF}_{\sigma, \mathcal{G}}\, T_{\mu-1}h(x_1, \ldots, x_n) = \sum_\beta d_\beta x_1^{\beta_1} \cdots x_n^{\beta_n}$

and find scalars $a_{\beta\alpha} \in \mathbb{C}$ such that $d_\beta = \sum_\alpha a_{\beta\alpha} c_\alpha$

4) For each β such that $d_\beta \neq 0$, return the operator

$$L_\beta = \sum_\alpha a_{\beta\alpha} D(\alpha_1, \ldots, \alpha_n)$$

Example

$I = (y^2, x^2 - y) \subset \mathbb{C}[x, y]$,

1) **Multiplicity** $\mu = 4$

2) **Taylor**

$$T_3 h(x,y) = c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{30}x^3 + c_{21}x^2y + c_{12}xy^2 + c_{03}y^3$$

3) **Normal Form**

$\mathrm{NF}(T_3 h) = [c_{00}] + [c_{10}]x + [c_{01} + c_{20}]y + [c_{11} + c_{30}]xy$

4) **Noetherian Operators**

$[c_{00}] \longrightarrow D(0,0) = 1$, $[c_{10}] \longrightarrow D(1,0) = \partial x$

$[c_{01} + c_{20}] \longrightarrow D(0,1) + D(2,0) = \partial y + \frac{1}{2}\partial x^2$

$[c_{11} + c_{30}] \longrightarrow D(1,1) + D(3,0) = \partial x \partial y + \frac{1}{6}\partial x^3$

Extension to zerodimensional modules

Example (from page 2)

1) $M = \langle xe_1 + e_2,\ xe_2 + ye_1,\ ye_2 \rangle$, $\mu(M) = 3$

$\mathcal{G} = \{xe_1 + e_2,\ xe_2 + ye_1,\ ye_2,\ y^2 e_1\}$

2) $T_2 v(x,y) = c^1_{00} e_1 + c^2_{00} e_2 + c^1_{10} x e_1 + c^2_{10} x e_2 + c^1_{01} y e_1 + c^2_{01} y e_2 + c^1_{20} x^2 e_1 + c^2_{20} x^2 e_2 + c^1_{11} xy e_1 + c^2_{11} xy e_2 + c^1_{02} y^2 e_1 + c^2_{02} y^2 e_2$

3) $\mathrm{NF}(T_2 v) = [c^1_{00}] e_1 + [c^2_{00} - c^1_{10}] e_2 + [c^1_{20} + c^1_{01} - c^2_{10}] y e_1$

4) operators: $\{(id, 0), (-\partial x, id), (\frac{1}{2}\partial x^2 + \partial y, -\partial x)\}$

Noether Normalization:

Consider an ideal $I \subset \mathbb{C}[z_1, \ldots, z_n]$ of dimension $d > 0$. There exists a *generic* linear change of coordinates

$$\varphi : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[x_1, \ldots, x_{n-d}, t_1, \ldots, t_d]$$ such that:

a) $\varphi(I) \cap \mathbb{C}[t_1, \ldots, t_d] = (0)$,
b) $\mathbb{C}[z_1, \ldots, z_n]/I$ is a finitely generated $\mathbb{C}[t_1, \ldots, t_d]$–module,
c) for each $i = 1 \ldots n - d$, $\varphi(I)$ contains a polynomial of the form

$$Q_i(t_1, \ldots, t_d, x_i) = x_i^{e_i} + p_1(t_1, \ldots, t_d)x_i^{e_i-1} + \cdots + p_{e_i}(t_1, \ldots, t_d)$$

Such change of coordinates can be described by a lower triangular matrix in $\mathrm{GL}_n(\mathbb{C})$.

Extension to higer dimension ideals

Using Noether Normalization:

$I = (x_1^2, x_2^2, x_1x_2, -x_1t + x_2) \in \mathbb{C}[x_1, x_2, t]$

1) $\mu(I) = 2$

2) $T_2h = c_{00} + c_{10}x_1 + c_{01}x_2$

$\longrightarrow \quad T_2\hat{h} = th = tc_{00} + tc_{10}x_1 + tc_{01}x_2$

3) $\mathrm{NF}(T_2\hat{h}) = [tc_{00}] + [c_{10} + tc_{01}]x_2$

4) operators: $\{1, \partial x + t\partial y\}$

References

D. Björk, *Ring of Differential Operators*, Birkhäuser, Boston, 1981.

A. Damiano, I. Sabadini, D. Struppa,
Computational Methods for the Construction of a Class of Noetherian Operators, Preprint.

L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Wiley Interscience, New York, 1970.

M.G. Marinari, H.M. Möller, T. Mora,
On Multiplicities of Polynomial System Solving, Trans. Am. Math. Soc., Volume **348** (1998), no. **8**.

U. Oberst, *The Construction Of Noetherian Operators*, J. Algebra **222** (1999), no. 2, 595–620.

V.P. Palamodov, *Linear Differential Operators with Constant Coefficients*, Springer, Berlin 1970.

B. Sturmfels, *Solving systems of polynomial equations.* CBMS Regional Conference Series in Mathematics, 97.