MID-ATLANTIC ALGEBRA CONFERENCE 2004

GMU, November 13-14, 2004

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Computation of Noetherian Operators

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Question 1

Given an ideal I in $R := \mathbb{C}[x_1, \ldots, x_n]$, how do we *test* if $\mathbf{f} \in \mathbf{I}$ without using a division algorithm?

Question 2

How do we describe the algebraic set $V(I)$ so that we take into account *geometric multiplicities*?

Nullstellensatz:

$$\{\text{algebraic sets of } \mathbb{C}^n\} \leftrightarrow \{\text{radical ideals of } \mathbb{C}[x_1, \ldots, x_n]\}$$

Question 3

How do we represent explicitly all the solutions to a system of linear constant coefficients *partial differential equations*?

Example 1:

$$\begin{cases} \frac{\partial f}{\partial x} & = & -g \\ \\ \frac{\partial f}{\partial y} & = & -\frac{\partial g}{\partial x} \\ \\ \frac{\partial g}{\partial y} & = & 0 \end{cases}$$

$\rightarrow$ in this case it could be done "by hand":

$f(x, y) = \frac{1}{2}ax^2 - bx + ay + c$

$g(x, y) = -ax + b$

Example 2: Regular functions $f : \mathbb{H}^n \longrightarrow \mathbb{H}$

$f = f_0 + if_1 + jf_2 + kf_3$, $q_i = x_{i0} + ix_{i1} + jx_{i2} + kx_{i3}$

$$\frac{\partial f(q_i)}{\partial \bar{q_i}} = 0 \Leftrightarrow \begin{cases} \frac{\partial f_0}{\partial x_{i0}} - \frac{\partial f_1}{\partial x_{i1}} - \frac{\partial f_2}{\partial x_{i2}} - \frac{\partial f_3}{\partial x_{i3}} & = 0 \\ \frac{\partial f_0}{\partial x_{i1}} + \frac{\partial f_1}{\partial x_{i0}} - \frac{\partial f_2}{\partial x_{i3}} + \frac{\partial f_3}{\partial x_{i2}} & = 0 \\ \frac{\partial f_0}{\partial x_{i2}} + \frac{\partial f_1}{\partial x_{i3}} + \frac{\partial f_2}{\partial x_{i0}} - \frac{\partial f_3}{\partial x_{i1}} & = 0 \\ \frac{\partial f_0}{\partial x_{i3}} - \frac{\partial f_1}{\partial x_{i2}} + \frac{\partial f_2}{\partial x_{i1}} + \frac{\partial f_3}{\partial x_{i0}} & = 0 \end{cases}$$

f is *regular* if $\frac{\partial f(q_1)}{\partial \bar{q_1}} = \cdots = \frac{\partial f(q_n)}{\partial \bar{q_n}} = 0$

Answer to Question i, $i = 1, 2, 3$:

Multiplicity Variety

Let V_j be algebraic varieties in $\mathbb{C}^n$ and let ∂_j be differential operators with polynomial coefficients, for $j = 1, ..., t$. Then we call

$$V = \{(V_1, \partial_1); (V_2, \partial_2); \ldots; (V_t, \partial_t)\}$$

a multiplicity variety.

The operators ∂_j's are called *noetherian operators*

Theorem Let I be an ideal of R. Then there exists a multiplicity variety V such that a polynomial f belongs to I if and only if $\partial_j f_{|V_j} = 0$ for every $j = 1, \ldots, t$.

Theorem (Fundamental Principle of Ehrenpreis - Palamodov, 1960) Let $p_1(D), \ldots, p_r(D)$ be linear constant coefficients partial differential operators in n variables. Then the multiplicity variety $\{(V_1, \partial_1); \ldots; (V_t, \partial_t)\}$ associated to $I = (p_1, \ldots, p_r)$ is such that every function $f \in \mathbb{C}^\infty(\mathbb{R}^n)$ satisfying

$$p_1(D)f = \ldots = p_r(D)f = 0$$

can be represented as

$$f(x) = \sum_{j=1}^{t} \int_{V_j} \partial_j(e^{ix\cdot z})d\nu_j(z), \qquad (1)$$

for suitable Radon measures $d\nu_j$.

The above (1) is an integral representation of the solutions of the system of equations "given" by $I = (p_1, \ldots, p_r)$. It generalizes the well known **Euler's formula**

How do we actually compute a multiplicity variety?

$$V = \{(V_1, \partial_1); (V_2, \partial_2); \ldots; (V_t, \partial_t)\}$$

Primary Decomposition: $I = Q_1 \cap \cdots \cap Q_t$ gives

$$V(I) = V(Q_1) \cup \cdots \cup V(Q_t)$$

so we can define $\mathbf{V}_j = V(Q_j)$ and then we just look for the ∂_j's to attach to each irreducible component.

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How do we compute the noetherian operators ∂_j's?

Case 1 (Euler's formula)

If $n = 1$ we have $I = (q(x)^\alpha)$ where $q(x) = x - a$ is irreducible. The operators in this case are simply

$$id, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}, \ldots, \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}$$

Example

Given the ODE $\frac{\partial^\alpha f}{\partial z} = 0$ its symbol is $q(x) = x^\alpha$ and using the integral formula (1) we get $f(z) =$

$$= c_0 (e^{xz})_{|x=0} + \cdots + c_{\alpha-1} \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}} (e^{xz})_{|x=0} =$$

$$= c_0 + c_1 z + c_2 z^2 + \cdots + c_{\alpha-1} z^{\alpha-1}$$

Case 2 (Principal Ideals)

If $n \geq 1$ we have $I = (q(x_1, \ldots, x_n)^\alpha)$, $q(x_1, \ldots, x_n)$ irreducible. Again

$$id, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial^{\alpha-1}}{\partial x_1^{\alpha-1}}$$

a long as x_1 appears as a simple power in q (*normal position*)

Case 3 (Wave Equation)

$$\frac{\partial^2}{\partial z^2}u(z,t) - \frac{\partial^2}{\partial t^2}u(z,t) = 0$$

$$p(x,y) = x^2 - y^2 = (x+y)(x-y)$$

$$\mathbf{V} = \{(x+y=0, id); (x-y=0, id)\}$$

$$u(z,t) = F(z+t) + G(z-t)$$

Case 4 (Zerodimensional Ideals)

$$t = x_1^{i_1} \cdots x_n^{i_n}, \quad D(t) = D(i_1, \dots, i_n) = \tfrac{1}{i_1! \cdots i_n!} \partial x_1^{i_1} \cdots \partial x_n^{i_n}$$

Space of differential operators:
$\mathcal{D} = \mathrm{Span}_{\mathbb{C}}(\{D(t)\}) = \mathbb{C}[\partial x_1, \dots, \partial x_n]$

Derivative-like morphisms σ_j:

$$\sigma_j\, D(i_1, \dots, i_n) = \begin{cases} D(\dots, i_j - 1, \dots) & \text{if } i_j > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Closed Subspaces

Consider $\mathcal{D}$ and extend σ_j to $\mathcal{D}$. A subspace U of $\mathcal{D}$ is *closed* if

$$\sigma_j(U) \subseteq U, \ \forall j$$

Correspondence $\Delta \longleftrightarrow \mathcal{I}$

Suppose that $V(I) = \{(0,\ldots,0)\}$:

$$\Delta(I) := \{L \in \mathcal{D} \,|\, L(f)(0,\ldots,0) = 0 \;\forall f \in I\}.$$

Similarly, we associate to each subset $U \subseteq \mathcal{D}$ an ideal

$$\mathcal{I}(U) := \{f \in R \,|\, L(f)(0,\ldots,0) = 0 \;\forall L \in U\}$$

Theorem Let $\mathfrak{m} = (x_1,\ldots,x_n)$. There is a **one to one** correspondence:

$$\{\mathfrak{m} - \mathrm{primary\,ideals\,in}\,R\} \underset{\mathcal{I}}{\overset{\Delta}{\leftrightarrows}} \{\mathrm{closed\,subspaces\,of}\,\mathcal{D}\}$$

Consequences:

Consider a primary zerodimensional ideal I centered at the origin with multiplicity μ:

- the noetherian operators associated to I are exactly $\mu = \dim_{\mathbb{C}}(\Delta(I))$

- the identity $id_{\mathcal{D}}$ is always a noetherian operator

- the maximum degree of an operator is $\mu - 1$

These facts and the use of Gröbner Bases techniques lead to an algorithm that can be easily implemented in CoCoA

Algorithm

input: a Gröbner Basis $\mathcal{G}$ of I **output**: $\Delta(I) = \{L_\beta\}$

1) Multiplicity of $\mu(I) = \dim_{\mathbb{C}}(R/I)$

2) Taylor expansion at the origin of a polynomial $h \in R$:

$$T_{\mu-1}h(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{N}^n,\, |\alpha| \leq \mu-1} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

3) Normal Form: $\mathrm{NF}_{\sigma,\mathcal{G}}\, T_{\mu-1}h(x_1, \ldots, x_n) = \sum_\beta d_\beta x_1^{\beta_1} \cdots x_n^{\beta_n}$

and find scalars $a_{\beta\alpha} \in \mathbb{C}$ such that $d_\beta = \sum_\alpha a_{\beta\alpha} c_\alpha$

4) For each β such that $d_\beta \neq 0$, return the operator

$$L_\beta = \sum_\alpha a_{\beta\alpha} D({x_1}^{\alpha_1} \cdots {x_n}^{\alpha_n})$$

Example

$I = (y^2, x^2 - y) \subset \mathbb{C}[x, y]$,

1) **Multiplicity** $\mu = 4$

2) **Taylor**

$$T_3h(x, y) = c_{00}+c_{10}x+c_{01}y+c_{20}x^2+c_{11}xy+c_{02}y^2+c_{30}x^3+c_{21}x^2y+c_{12}xy^2+c_{03}y^3$$

3) **Normal Form**

$\mathrm{NF}(T_3h) = [c_{00}] + [c_{10}]x + [c_{01} + c_{20}]y + [c_{11} + c_{30}]xy$

4) **Noetherian Operators**

$[c_{00}] \to D(0, 0) = 1$, $[c_{10}] \to D(1, 0) = \partial x$

$[c_{01} + c_{20}] \to D(0, 1) + D(2, 0) = \partial y + \frac{1}{2}\partial x^2$

$[c_{11} + c_{30}] \to D(1, 1) + D(3, 0) = \partial x \partial y + \frac{1}{6}\partial x^3$

Extension to zerodimensional modules

Example (from page 2)

1) $M = \langle xe_1 + e_2,\, xe_2 + ye_1,\, ye_2 \rangle$, $\mu(M) = 3$

$\mathcal{G} = \{xe_1 + e_2,\, xe_2 + ye_1,\, ye_2,\, y^2 e_1\}$

2) $T_2 v(x,y) = c^1_{00} e_1 + c^2_{00} e_2 + c^1_{10} x e_1 + c^2_{10} x e_2 + c^1_{01} y e_1 + c^2_{01} y e_2 + c^1_{20} x^2 e_1 + c^2_{20} x^2 e_2 + c^1_{11} xy e_1 + c^2_{11} xy e_2 + c^1_{02} y^2 e_1 + c^2_{02} y^2 e_2$

3) $\mathrm{NF}(T_2 v) = [c^1_{00}] e_1 + [c^2_{00} - c^1_{10}] e_2 + [c^1_{20} + c^1_{01} - c^2_{10}] y e_1$

4) operators: $\{(id, 0), (-\partial x, id), (\frac{1}{2}\partial x^2 + \partial y, -\partial x)\}$

Extension to higer dimension ideals

Using Noether Normalization:

$I = (x_1^2, x_2^2, x_1x_2, -x_1t + x_2) \in \mathbb{C}[x_1, x_2, t]$

1) $\mu(I) = 2$

2) $T_2h = c_{00} + c_{10}x_1 + c_{01}x_2$

$\rightarrow \quad T_2\hat{h} = th = tc_{00} + tc_{10}x_1 + tc_{01}x_2$

3) $\mathrm{NF}(T_2\hat{h}) = [tc_{00}] + [c_{10} + tc_{01}]x_2$

4) operators: $\{1, \partial x + t\partial y\}$

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