Syzygies of multi-variable higher spin Dirac operators on $\mathbb{R}^3$

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Abstract. This paper is a short report on the generalization of some results of our previous paper [12] to the case of spin $j/2$ Dirac operators in real dimension three for arbitrary odd integer j. We use an explicit formula for the local expression of such operators to study their algebraic properties, construct the compatibility conditions of the overdetermined system associated to the operator in several spatial variables, and we prove that its associated algebraic complex, dual do the BGG sequence coming from representation theory, has substantially the same pattern as the Cauchy-Fueter complex.

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1. Introduction

The study of overdetermined systems of linear constant coefficient partial differential operators arises naturally when we consider some problems in Clifford Analysis. The Dirac operator on a general Spin manifold of dimension m is, with obvious meaning of symbols,

$$D = \sum_{i=1}^{m} e_i \cdot \nabla_{e_i}$$

and its expression in local coordinates corresponds to the classical Dirac operator

$$\partial_{\mathbf{x}} = \sum_{i=1}^{m} e_i \frac{\partial}{\partial x_i}$$

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which is nothing but a hypercomplex version of the Cauchy-Riemann operator $\partial_{\bar{z}} = \partial_x + i\partial_y$ for functions of two real variables x and y. Such operator is elliptic and so it is clearly surjective on the space of, for instance, smooth functions on open convex sets. The question of its surjectivity becomes less trivial when considering functions of several vector variables $\mathbf{x_i} = (x_{i1}, \ldots, x_{im})$, $i = 1, \ldots, k$. In this case we need to solve a non-homogenous system of equations

$$\begin{cases} \partial_{\mathbf{x_1}} f &= & g_1 \\ \partial_{\mathbf{x_2}} f &= & g_2 \\ & \cdots & \\ \partial_{\mathbf{x_k}} f &= & g_k \end{cases} \tag{1.1}$$

in which each $\partial_{\mathbf{x_i}}$ is the Dirac operator written using the variables $\mathbf{x_i}$, and where f and g_i are in a suitable space of generalized functions. Until now, a complete answer to this question, i.e. a closed form for the conditions that $g_1, \ldots, g_k$ have to satisfy, for all number of variables k and for any dimension m, is not known. The are basically two different approaches to this task. The first, whose methodology is presented in [11], uses Gröbner bases tools to encode the system (1.1) (or any other overdetermined system of linear constant coefficient PDEs) into a polynomial module $\mathcal{M}_{k,m}$, so to construct the first module of syzygies. This describe the compatibility conditions of the system explicitly. Iterating this process one arrives to a free resolution (see Section 3 for definitions and notations)

$$0 \longrightarrow R^{\beta_t} \longrightarrow R^{\beta_{t-1}} \longrightarrow \ldots \longrightarrow R^{\beta_0} \longrightarrow \mathcal{M}_{m,k} \longrightarrow 0 \tag{1.2}$$

where $R = \mathbb{C}[x_{ij}]$ is the polynomial ring over all the variables and β_i is a positive integer. In the case $k = 3$ this method has produced the results of [23] in which all the maps in the resolution are described explicitly. An alternative approach comes from the study of homogeneous spaces and parabolic geometries, together with some considerations on the symmetry of the operator [16, 25]. By identifying the symmetries of the operator $D_{(k)} = [\partial_{\mathbf{x_1}} \cdots \partial_{\mathbf{x_k}}]^t$ (i.e. the action of a group with respect to which the operator is invariant) and by considering the functions f and g_i, $i = 1, \ldots, k$ as taking values in opportune irreducible representation spaces of such group, it is possible (using the Weyl orbit on weight spaces) to associate to the operator a sequence of irreducible representation spaces $\mathbb{V}_j$ and operators $E_j : \mathcal{C}^\infty(\mathbb{V}_j) \longrightarrow \mathcal{C}^\infty(\mathbb{V}_{j+1})$ which then share the same symmetry of the operator $D_{(k)} =: E_0$. In this case the first space is simply the spinor space $\mathbb{V}_0 = \mathbb{S}_{1/2}$ and the target is the space of spinor valued forms $\mathbb{V}_1 = \mathbb{S}_{1/2} \otimes (\mathbb{R}^m)^*$. The advantage of the first approach is that it is possible to use a computer to run an algorithm for the construction of the resolution in every specific case (at least for small values of k and m see [13]). By taking the dual of (1.2) we obtain a complex of free modules whose maps are the Fourier transforms of some differential operators. So the construction of a sequence of operators is explicit and the result is a complex of differential operators. However, the invariance of the original operator is lost since the computation of the resolution, which relies on Buchberger's algorithm for the calculation of Gröbner bases, does not make use of any possible symmetry of the module $\mathcal{M}_{k,m}$ (note that Gröbner basis calculations rely essentially on the combinatorial of monomial ideals and modules, the construction of which is based upon a notion of ordering on the space of monomials, which excludes any possibily of taking symmetry into account). On the other hand, using the geometric approach, we obtain a sequence of operators that posses the same symmetry as the initial operator, but we do not know a priori whether the sequence is a complex. Recent joint research with the aim of comparing the two methods has shown, however, that for the case of several Dirac operators in dimension four, which leads after some identifications to the case of functions of the Cauchy-Fueter operator acting on several quaternionic variables, such approaches produce in fact the same result [4, 10]. Furthermore, the recent work of [16] revealed that the two approaches also coincide in the case of 2 operators in dimension at least 6 (a stable behavior of the complex is reached when $m \geq 2k$ as conjectured in [11] based in the results of [21] and the validity of the Fisher decomposition in several variables). Some additional consideration on the geometric nature of the system 1.1 for $k = 2$ using the tool of Penrose transform can be also found in [20].

In this paper we focus our attention on the case of real dimension 3, and we generalize the Dirac operator to the so called higher spin Dirac operators. We will not follow the geometric approach but we will limit ourself to using computational algebraic analysis techniques. Many first order differential operators arise naturally in representation theory as projections onto irreducible modules of the covariant derivative acting on the space of functions with values in suitable representation of a Lie group (see [25]). In our case, we will consider the (irreducible) representations $S_{j/2}$ of $Spin(3) \simeq SU(2)$, as defined in the next section. The case $j = 1$ is nothing but the classical Dirac operator, while the case $j = 3$ is generally called Rarita-Schwinger operator. Properties of nullsolutions of the latter on a flat space and on the sphere are described in [3] and have then been generalized to the case of higher spin operators in [5, 6] using Clifford analysis. See also [2] for an introduction to higher spin operators and for a summary of their properties in conformal geometries. In particular, it is proved in [6] that such operators, clearly elliptic when $j = 2i - 1$, factor the i-th power of the Laplacian. For the case of the Dirac operator, the fact that it factors the Laplacian results in the fact that the system (1.1) has quadratic syzygies [23]. Therefore one may think that for the higher spin operators the first syzygies should have higher degree. We already proved in our previous paper [12] that this is not the case for the Rarita-Schwinger operator. In this paper we extend such result to any value of j. The main goal of this paper is in fact to prove the more general statement, analogous to Theorem 3.1 of [12], that the free resolution of the module associated to several higher spin operators in dimension three is essentially the same as the one for the case $j = 1$, which in turns is tightly related to the one for the Moisil-Theodorescu operator [22] and for the Cauchy-Fueter operator [1].

2. A local formula for the operators

Let us consider the complex irreducible representations $S_{j/2}$ of the group $Spin(3,\mathbb{C})$. Usually one sees the spinor space $S_{1/2}$ as a minimal left ideal in the clifford algebra $\mathbb{C}_n$ and then realizes higher spin representation by decomposing recursively the tensor product $S_{(j+2)/2} \otimes \mathbb{R}^n$ into irreducibles Spin modules. However for our purpose, since we work in dimension 3 and given that the group $Spin(3)$ is isomorphic to $SU(2)$, $S_{j/2}$ can be thought of the space of univariate complex polynomials of degree at most j with basis $\{1, z, \ldots, z^j\}$, with the action of an element $g = \binom{a\,b}{c\,d} \in SU(2)$ on z^i defined as as $gz^i = (bz+d)^{j-i}(az+c)^i$. The subscript $j/2$ in this case denotes the first value of the highest weight $(\frac{j}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ of the representation. Consider the space $\mathbb{S}_{j/2}$ of functions $f : U \to S_{j/2}$ where U is an open set of $\mathbb{R}^3$. Consider now the representation $(\mathbb{R}^3)^*$, dual to the defining representation of $Spin(3)$ and take the space of spinor valued one-forms $\Lambda^1(\mathbb{S}_{j/2}) := \mathbb{S}_{j/2} \otimes (\mathbb{R}^3)^*$. The usual covariant derivative maps scalar spinor functions to one-forms, and since the latter space is decomposable into irreducibles, it is possible to compose it with the invariant projection on the irreducible component isomorphic to $\mathbb{S}_{j/2}$:

$$\mathbb{S}_{j/2} \xrightarrow{\nabla} \Lambda^1(\mathbb{S}_{j/2}) \xrightarrow{\pi_j} \mathbb{S}_{j/2}.$$

Definition 2.1. With the above notation, we call the operator

$$\mathcal{D}_j := \pi_j \circ \nabla$$

the higher spin Dirac operator with spin $j/2$.

Remark 2.2. First, the invariant projections are defined up to a constant factor, so our operators are all defined up to this constant. This obviously does not affect the analysis of the space of nullsolutions. Second, note that the above definition actually works for any dimension and could be given in the more general setting of spin manifolds, but since we are interested in the local properties of the operators we will only deal with the flat case.

For a thorough description of many algebraic properties of such operators see [17]. In particular we will borrow from this paper the explicit local formula 3.1:

$$\mathcal{D}_j = \sum_{i=1}^{3} \rho_j(e_i) \cdot \frac{\partial}{\partial x_i} \tag{2.1}$$

where e_i is the canonical basis element of $\mathbb{R}^3$ and x_i a local coordinate, and the morphisms ρ_j are given by

$$\begin{cases} \rho_j(e_1)z^k &= i(2k-j)z^k \\ \rho_j(e_2+ie_3)z^k &= -2kz^{k-1} \\ \rho_j(e_2-ie_3)z^k &= 2(j-k)z^{k+1} \end{cases} \qquad \text{with } k = 1, \ldots, j. \tag{2.2}$$

Each operator $\mathcal{D}_j$ can then be represented, with respect to the basis introduced above, as a square $m+1$ matrix in which the entries are polynomials of degree one in $\mathbb{C}[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}]$. By taking the Fourier transform up to a constant $-\sqrt{-1}$ and by using the change of coordinates

$$X = -ix_1, \qquad Y = x_2 + ix_3, \qquad Z = -x_2 + ix_3,$$

(see [12] for discussion) we get a square matrix P_j with entries in $\mathbb{C}[X, Y, Z]$:

$$P_j = \begin{pmatrix} jX & Z & 0 & 0 & \dots & 0 & 0 \\ jY & (j-2)X & 2Z & 0 & \dots & 0 & 0 \\ 0 & (j-1)Y & (j-4)X & 3Z & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 3Y & (4-j)X & (j-1)Z & 0 \\ 0 & 0 & \dots & 0 & 2Y & (2-j)X & jZ \\ 0 & 0 & \dots & 0 & 0 & Y & -jX \end{pmatrix}. \tag{2.3}$$

The matrix P_j is then tri-diagonal with multiples of X on the main diagonal, multiples of Z in the first upper off-diagonal and multiples of Y in the lower off-diagonal. It is immediate to see that

Proposition 2.3. *The operator $\mathcal{D}_j$ is elliptic if and only if j is odd. Moreover, for j odd the operator $\mathcal{D}_j$ factorizes the operator $\Delta^{\frac{j+1}{2}}$ where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplace operator in $\mathbb{R}^3$.*

Proof. The symbol of $\mathcal{D}_j$ is P_j and its determinant is easily calculated, by induction on j or simply by applying the definition of determinant and noticing that only transpositions of two column indices survive in the sum, as

$$\mathrm{Det}(P_j) = (\prod_{k=0}^{j}(j-2k)) \cdot (X^2 + YZ)^{\frac{j+1}{2}} = (\prod_{k=0}^{j}(2k-j)) \cdot (x_1^2 + x_2^2 + x_3^2)^{\frac{j+1}{2}}$$

so it is zero if and only if j is even. The proof of the fact that D_j factorizes a power of the Laplacian can be found in [6]. Note also that it is possible to prove it using a computer algebra system such as CoCoA [7]. We skip the details here for the sake of brevity but more details can be found on our webpage [14] where we present some explicit calculations. □

3. Higher spin operators in several variables

Because of Proposition 2.3, we will now only consider the operators $\mathcal{D}_j$ with j odd. Let us now recall some of the definitions for the algebraic analysis of linear constant differential operators. We refer the reader to [11] for a complete treatment Let $R = \mathbb{C}[X_1, \dots, X_m]$ be a ring of polynomials and let $D = -\sqrt{-1}(\partial_1, \dots, \partial_m)$ where ∂_i actually means $\frac{\partial}{\partial X_i}$. Let P be an element of $Mat_{\beta_1, \beta_0}(R)$, i.e. a rectangular β_1 times β_0 matrix with polynomial entries, and consider its fourier transform $P(D)$ (up to a constant). Then $P(D)$ is called a linear constant coefficient partial differential operator. Let f be a smooth function of m real variables with values in $\mathbb{R}^{\beta_0}$ or $\mathbb{C}^{\beta_0}$. Then we can consider the homogeneous system of PDE $P(D)f = 0$ or its non homogeneous version

$$P(D)f = g, \tag{3.1}$$

where g is again a smooth function of m real variables with values in a space of dimension β_1. The main algebraic object associated to 3.1 is the polynomial module given by the cokernel of the map $P^t : R^{\beta_1} \to R^{\beta_0}$, namely $\mathcal{M} = R^{\beta_0}/P^t R^{\beta_1}$. This is the quotient of the free module R^{β_0} with the module generated by the row vectors of the matrix P (by convention we identify a polynomial matrix with its column span). It is a fundamental theorem of commutative algebra that

Theorem 3.1 (Hilbert Syzygy Theorem). *Every finitely generated R-module $\mathcal{M}$ has a finite free resolution, i.e. there exists a sequence of polynomial maps between free modules*

$$\mathcal{F}: \qquad 0 \to R^{\beta_s} \xrightarrow{\varphi_{s-1}} R^{\beta_{s-1}} \xrightarrow{\varphi_{s-2}} \dots \xrightarrow{\varphi_1} R^{\beta_1} \xrightarrow{\varphi_0} R^{\beta_0} \to 0 \tag{3.2}$$

which is exact, except at the zeroth spot where the homology is $M \simeq R^{\beta_0}/\mathrm{im}(\varphi_0)$.

A sequence like 3.2 is far from being unique. However its constuction is clear and simple to describe: one starts with a map φ_0 which sends the basis elements of R_1^{β} to the β_1 generators of $\mathcal{M}$. Then one calculates the kernel of such maps, thing that can be done explicitely using Gröbner basis theory [18], and chooses a set of β_2 generator for such kernel. Iterating this process, the whole sequence of maps is constructed explicitly. A bound on the length of $\mathcal{F}$ is given by $s \leq m$, but since at each step there is an arbitrary choice of generators of a module to be made, the result is that $\mathcal{F}$ is far from being unique. However, if one starts with a *graded module* (see [19] for precise definitions), so that essentially there exists a system of homogeneous (w.r.t. a given grading on R) generators for M, then it is possible to extract at each step a minimal set of generators for the kernels of the maps involved. This minimality makes the positive integers β_i well defined, and one can show that any two such minimal free resolutions are isomorphic as complexes. The integers β_i are then called Betti numbers. The module $im(\varphi_i)$ in a minimal free resolution is called the i-th *syzygy module* associated to $\mathcal{M}$. If one now dualizes the sequence 3.2 by applying the functor $\mathrm{Hom}(\,\cdot\,, R)$, one obtains a complex

$$\mathcal{F}^*: \qquad 0 \to R^{\beta_0} \xrightarrow{\varphi_0^*} R^{\beta_1} \xrightarrow{\varphi_1^*} \dots \xrightarrow{\varphi_{s-2}} R^{\beta_{s-1}} \xrightarrow{\varphi_{s-1}} R^{\beta_s} \to 0 \tag{3.3}$$

in which all the maps φ_i^* are the transpose of the maps in $\mathcal{F}$. When starting with the module $\mathcal{M}$ associated to a differential operator $P(D)$, the complex above contains a lot of information on the algebraic nature of the system 3.1 and constitutes an analogue of the so called *BGG* sequence which starts with the operator $P(D)$ (provided that it possesses a given symmetry). In particular, it is possible to extract from 3.3 the following information:

(i) *the compatibility conditions of the system 3.1*

(ii) *properties of removability of compact singularities for the nullsolutions of* $P(D)$

(iii) *a description of the spaces of hyperfunctions associated to the operator*

(iv) *the dimensions of the spaces of its homogeneous polynomial solutions, with explicit formulas*

The first item (i) in the above list is a very important information on the system of equations one wants to study. Basically, in order for $P(D)f = g$ to be compatible, there are some conditions on the data function g that has to be satisfied. Those correspond to a differential operator, say $P_1(D)$, such that

$$P_1(D)g = 0. \tag{3.4}$$

It is clear that if one finds a matrix of polynomials P_1 such that $P_1 * P$ is the zero matrix, then the equation 3.4 is a necessary condition in order for a solution to 3.1 exists. It turns out that the map φ_1^* in $\mathcal{F}^*$ represents exactly such conditions, and that they are actually necessary and sufficient conditions, on open convex domains. Iteratively, the map φ_2^* represents the compatibility conditions of the inhomogeneous system $P_1(D)g = h$ and so on.
As far as the removability of compact singularity, which we will study for the higher spin Dirac operators in section 4.3, one is concerned with the exactness of the complex 3.3. Dualizing an exact complex like 3.2, in fact, does not necessarily produce another exact complex. Exactness is measured by the modules

$$Ext^i_R(\mathcal{M}, R)$$

which constitutes the cohomology modules for $\mathcal{F}$. vanishing of such modules results in nice regularity properties of the kernel of $P(D)$ on spaces of generalized functions. For details, see again [11]. We will study the vanishing of the first cohomology module which is associated to the so called Hartogs phenomenon. Item (iii) of the list will not be treated in this paper, but we refer the reader to the work of [9, 24] for the case of the Cauchy-Fueter operator in one and two quaternionic variables.
We are now ready to approach the study of the higher spin Dirac operators. We will first analyze the case of one vector variable ($k = 1$) and then go to the multivariable case.

Let us consider the module $\mathcal{M}_j := R^{j+1}/\mathrm{Im}(P_j^t)$ where $R := \mathbb{C}[X, Y, Z]$ and P_j is defined in 2.3. This is the module associated to $\mathcal{D}_j$. It follows immediately from Proposition 2.2 of [8], since P_j in this case is a square non-singular matrix due to proposition 2.3, that for the case of just one spatial variable, the module $\mathcal{M}_j$ has free resolution

$$0 \longrightarrow R^{j+1} \xrightarrow{P_j^t} R^{j+1} \longrightarrow 0$$

which in turn means that the operator $\mathcal{D}_j$ is surjective on the space of smooth functions defined on an open convex set of $\mathbb{R}^3$. The first cohomology of the free resolution is $Ext^1_R(\mathcal{M}_j, R) = R^{j+1}/im(P_j)$, fact that we will use in proposition 4.3 to prove that solution to this operator can have compact singularities, exactly as monogenic functions in one vector variable do, being essentially a generalization of holomorphic function.
The interesting case to consider is then when f is a function of several spatial variables X_i, Y_i, Z_i, $i = 1, \dots, k$, and when $\mathcal{D}_{ji}$ is the higher spin operator with respect to the three variables X_i, Y_i, Z_i, i.e. the differential operator whose symbol is

$$P_{ji} = \begin{pmatrix} jX_i & Z_i & 0 & 0 & \dots & 0 & 0 \\ jY_i & (j-2)X_i & 2Z_i & 0 & \dots & 0 & 0 \\ 0 & (j-1)Y & (j-4)X_i & 3Z_i & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 3Y_i & (4-j)X_i & (j-1)Z_i & 0 \\ 0 & 0 & \dots & 0 & 2Y_i & (2-j)X_i & jZ_i \\ 0 & 0 & \dots & 0 & 0 & Y_i & -jX_i \\ . \end{pmatrix}. \tag{3.5}$$

We then consider the system, with obvious meaning of symbols,

$$\begin{cases} \mathcal{D}_{j1}f &= g_1 \\ \mathcal{D}_{j2}f &= g_2 \\ & \dots \\ \mathcal{D}_{jk}f &= g_k \end{cases} \tag{3.6}$$

which is the generalization of (1.1) to the case of spin $j/2$. The system can be written in the form $P_{(jk)}(D)f = g$ where $D = (\frac{\partial}{\partial X_1}, \frac{\partial}{\partial Y_1}, \frac{\partial}{\partial Z_1}, \dots, \frac{\partial}{\partial X_k}, \frac{\partial}{\partial Y_k}, \frac{\partial}{\partial Z_k})$ and $P_{(jk)}$ is the block matrix whose blocks are the matrices $P_{j1}, \dots, P_{jk}$ arranged in a column. The main abject of our study will be the finitely generated R-module

$$\mathcal{M}_{jk} := R^{j+1}/\mathrm{Im}(P^t_{(jk)}).$$

We know from [22] that the free resolution associated to $\mathcal{M}_{1k}$, which represents the Moisil-Theodorescu operator in k spatial variables, is

$$0 \longrightarrow R^{2(k-1)}_{(-2k)} \longrightarrow \dots \longrightarrow R^{\binom{2n}{3}}_{(-3)} \longrightarrow R^{2k}_{(-1)} \stackrel{P^t_{(1k)}}{\longrightarrow} R^2 \longrightarrow 0 \tag{3.7}$$

where the term $R_{(-n)}$ denotes a shift in the grading (the constants in $R_{(-n)}$ have degree n). In particular, the difference between two consecutive shifts gives the degree of the map between such spaces, so shifts are indicated as a mean to keep track of the degrees of the syzygies at each step. In this case, the last map to the right has degree 1, the second to last has degree 2, the third has degree 1 and so on. We proved in [12] that a similar result holds for $j = 3$ as the Rarita-Schwinger resolution is

$$0 \longrightarrow R^{4(k-1)}_{(-2k)} \longrightarrow \dots \longrightarrow R^{2\binom{2n}{3}}_{(-3)} \longrightarrow R^{4k}_{(-1)} \stackrel{P^t_{(3k)}}{\longrightarrow} R^4 \longrightarrow 0. \tag{3.8}$$

Up to a constant in the dimensions of the free modules (called Betti numbers), the resolutions look exactly the same. It seems then natural to suppose that for odd $j > 3$ we still observe a similar behavior in the Betti numbers. This is in fact the statement of our main result

Theorem 3.2. *Let j and k be positive integers with j odd. Let $R = \mathbb{C}[X_i, Y_i, Z_i \,|\, i = 1..k]$ be a ring of polynomials and let $P_{(jk)} = [P^t_{j1}, \ldots, P^t_{jk}]^t$ be the block matrix with entries in R defined above, i.e. the symbol of the spin $j/2$ Dirac operator in k spatial variables. Let $\mathcal{M}_{jk}$ be the cokernel of $P_{(jk)}$. Then the free resolution associated to $\mathcal{M}_{jk}$ is*

$$0 \to R^{\beta_{2k-1}}_{(-2k)} \longrightarrow \ldots \longrightarrow R^{\beta_3}_{(-4)} \longrightarrow R^{\beta_2}_{(-3)} \longrightarrow R^{k(j+1)}_{(-1)} \xrightarrow{P^t_{(jk)}} R^{(j+1)} \to 0. \tag{3.9}$$

where

$$\beta_d = k(j+1)\binom{2k-1}{d}\frac{d-1}{d+1}, \quad d > 1.$$

In particular, the length of the resolution is $2k-1$, the map at the second step from the right is quadratic, and all the other ones are linear.

Proof. We sketch here the proof of this statement. We do not give all the details because it is essentially the same proof given for the case of the Rarita–Schwinger operator in several variables, with a few modifications, and it utilizes a well known argument that has been applied successfully several times since the study of the multi-variable Cauchy-Fueter operator was carried out in [1]. Let $N_{jk} := \mathrm{Im}(P^t_{(jk)})$. First one calculates the monomial module $\mathrm{LT}_{\texttt{lexpos}}(N_{jk})$. It clearly contains all the leading terms of the row vectors of the matrix $P_{(jk)}$, namely $X_i e_t$ for $i = 1..k$ and e_t a basis element of R^{j+1}. It also contains the quadratic leading terms $Y_a Z_b e_t$ for e_t as before and $1 \leq a < b \leq k$, obtained as leading terms of the S-polynomials (see again [18] for definitions) of two corresponding rows in P_{ja} and P_{jb}, i.e. two rows which can be obtained from one another by just changing a into b (the other rows clearly do not have nonzero S-polynomials because their leading terms have different position e_t). The leading term module clearly does not contain other (minimal) generators because we only have 3 letters at our disposal so by iterating Buchberger's algorithms to find a Gröbner basis, all the new S-polynomials would have leading terms that are multiples of the one already calculated. So the leading term module is diagonal of the type $\mathrm{LT}(N_{jk}) = \oplus_{t=1}^{j+1} I_k e_t$ and $I_k := \langle\{X_i,\, Y_a Z_b \,|\, i = 1\ldots k, 1 \leq a < b \leq k\}\rangle$. This allows immediately to find the Hilbert series of $\mathcal{M}_{jk}$:

$$\mathcal{H}_{\mathcal{M}_{jk}}(z) = (j+1)\mathcal{H}_{R/I_k}(z) = (j+1)\frac{1+(k-1)z}{(1-z)^{2k-1}}.$$

In general it is not enough to know the Hilbert series of a module in order to find its Betti numbers. However we can prove that the module $\mathcal{M}_{jk}$ has Castelnuovo-Mumford regularity two by noticing that

$$\{Y_1,\, Z_k,\, Z_1 + Y_2, \ldots,\, Z_{k-1} + Y_k\}$$

is a regular sequence of length $k+1$ for $\mathcal{M}_{jk}$, so it follows that the resolution is "pure", i.e. at each step the syzygies only contain elements of the same degree (indeed linear at the first step, quadratic for the first syzygies, and then linear, see again [1] for details). This is a case in which the Betti numbers can be then calculated directly from $\mathcal{H}(z)$ as the coefficients, up to sign, of the numerator $N(z)$ of the non-reduced Hilbert series $\mathcal{H}_{\mathcal{M}_{jk}}(z) = \frac{N(z)}{(1-z)^{3k}}$. □

4. Consequences

We list here some facts that can be deduced directly from Theorem 3.2 and some simple considerations on the Hilbert series of the module $\mathcal{M}_{jk}$. In particular we are interested in the explicit form of the compatibility conditions of the system (3.6), which correspond to the quadratic map of (3.9). We are also able to calculate the dimensions of the space of homogeneous polynomial solutions, and finally we study the exactness of the complex obtained as the dual of (3.9).

4.1. Quadratic syzygies

The fact that the first syzygies are only quadratic constitutes a piece of information that is definitely new in the analysis of the higher spin Dirac operators. Consider for example the case $k=2$ and the operators whose symbols are P_{j1} and P_{j2}. If we denote by $L_{ji} = (X_i^2 + Y_i Z_i)^{\frac{j+1}{2}} I$ the symbol of the $\frac{j+1}{2}$-th power of the Laplace operator, then there exists a polynomial matrix A_{ji} with entries in $\mathbb{C}[X_i, Y_i, Z_i]$ such that $A_{ji}P_{ji} = L_{ji}$ for $i = 1,2$ and j odd, according to[6]. It is then easy to construct two very natural syzygies of P_{j1} and P_{j2} as follows:

$$(P_{j2}A_{j1})P_{j1} - (L_{j1})P_{j2} = 0$$

$$(L_{j2})P_{j1} - (P_{j1}A_{j2})P_{j2} = 0$$

using that L_{ji} commutes with every square matrix. However, since $P_{j2}A_{j1}$, $P_{j1}A_{j2}$ and L_{ji}, $i = 1,2$ are homogeneous matrices of degree $j+1$, the above relations are far from being quadratic (except of course for $j=1$). In fact, one could prove in the general case of k operators (see again [14] for explicit examples) that there exist linear matrices $\widetilde{P_{ji}}$ such that the generalized commutator

$$C_{rs} := \widetilde{P_{jr}} P_{js} - \widetilde{P_{js}} P_{jr}, \quad r,s = 1..k,\ r \neq s$$

is diagonal. This allows to write explicitly all syzygies of the type

$$(C_{rs} + P_{jr}\widetilde{P_{js}})P_{jr} - (P_{jr}\widetilde{P_{jr}})P_{js} = 0.$$

Remark 4.1. The operator associated to $\widetilde{P_{ji}}$ plays the role of a conjugate operator, like ∂_{q_i} does for the case of the Cauchy-Fueter operator $\partial_{\bar{q}_i}$ and $-\partial_{\mathbf{x_i}}$ for the case of the classical Dirac.

4.2. Polynomial solutions

The algebraic analysis of the module $\mathcal{M}_{jk}$ is tightly related to the study of polynomial solutions of the system of differential equations it represents. In particular we consider the Hilbert function, i.e. the numerical function $h(d) = \dim_{\mathbb{C}}([\mathcal{M}_{jk}]_d)$, $d \in \mathbb{N}$, where $[\_\_]_d$ denotes the homogeneous component of degree d of an $\mathbb{N}$-graded module. Its generating function is the Hilbert series by definition. The Hilbert function gives also the dimension of the space of polynomial solutions of degree exactly d (see [11] and Proposition 4.1 of [12]). It is then easy to prove directly the following

Corollary 4.2. *Let $\mathcal{P}$ be the space of polynomial solutions to the system of equations* (3.6). *Then*

$$\dim_{\mathbb{C}}([\mathcal{P}]_d) = (j+1)(d+1)\binom{d+k-1}{k-1}.$$

Proof. It suffices to calculate $h(d)$ as the d-th Taylor coefficient of the power series for $\mathcal{H}_{\mathcal{M}_{jk}}(z)$. □

4.3. Hartogs phenomenon

It is a well known fact in (hyper) complex analysis that (hyper) holomorphic functions of several variables cannot have compact singularities, while functions of just one variable can. It was known to Hartogs since 1906 that holomorphic functions of several complex variables satisfy this "rigidness" property. However it was one of the first results in modern algebraic analysis, due to Ehrenpreis, and one of the first applications of this theory to some classical example of constant coefficient differential operators, that Hartogs' theorem can be extended to a larger class of functions, not just holomorphic ones. We want to prove that it is possible to show an analogue of such result for the nullsolutions of the higher spin Dirac operator in several variables. According to [11] this is equivalent to the vanishing of the module $\mathrm{Ext}^1_R(\mathcal{M}_{jk}, R)$ which is the first cohomology of the module associated to the operator.

Proposition 4.3. *Let $\mathcal{M}_{jk}$ be the module associated to the $j/2$-spin Dirac operator in k vector variables, with j odd. Then*

$$\mathrm{Ext}^1_R(\mathcal{M}_{jk}, R) = 0 \quad \text{if and only if} \quad k > 1.$$

Proof. Since the free resolution associated to the module $\mathcal{M}_{j1}$ ends at the first step, its first cohomology is simply $R^{j+1}/\mathrm{Im}P_{(j1)}$ so it cannot vanish. For $k = 2$ it is easy to prove the result using CoCoA, either calculating directly the first Ext module or by using the fact that such module is zero if and only if the maximal minors of the matrix $P_{(j2)}$ are coprime, which can be easily checked directly. It then follows that the minors for $k > 2$ are coprime as well since they contain the minors for $k = 2$ as a subset. □

Corollary 4.4. *Let f be a solution to the homogeneous version of the system* (3.6) *defined on $U \setminus K \subseteq (\mathbb{R}^3)^k$ with U open and K compact. Then there exists a unique* *extension* $\widetilde{f}$ *of* f *to* U *which is in the kernel of the same operator if and only if* $k > 1$.

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