MEGA 2005

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Computation of Dirac syzygies using MEGAforms

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OUTLINE

1. The **Dirac complex**: a commutative approach

2. An idea inspired by the **De Rham complex**

3. **Megaforms** and **radial relations**

4. The calculation of Dirac **syzygies** using CoCoA

5. Conclusions

DIRAC COMPLEX

Clifford algebra : $\mathcal{C}_n = \langle e_1, \ldots e_n \mid e_i e_j + e_j e_i = -2\delta_{ij},\ i, j = 1 \ldots n \rangle$

real variables $\underline{x}_1, \ldots, \underline{x}_k \in \mathbb{R}^n, \quad \underline{x}_i = (x_{i1}, \ldots, x_{in}),\ i = 1 \ldots k$

differentiable functions $f : (\mathbb{R}^n)^k \longrightarrow \mathcal{C}_n$

Dirac operator $\partial_{\underline{x}_i} = \sum_j e_j \frac{\partial}{\partial x_{ij}},\ i = 1 \ldots k$

Dirac System defining **monogenic** functions $\begin{cases} \partial_{\underline{x}_1} f & = & 0 \\ \vdots & & \\ \partial_{\underline{x}_k} f & = & 0 \end{cases}$

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ALGEBRAIC ANALYSIS of the Dirac system, using

$R = \mathbb{C}[x_{11}, \ldots, x_{kn}]$

$D = -i(\frac{\partial}{x_{11}}, \ldots, \frac{\partial}{x_{kn}})$

$P \in \mathrm{Mat}_{(k \cdot 2^n) \times 2^n}(R)$ **symbol** matrix

compact form of the Dirac system: $P(D)\vec{f} = \vec{g}$

we study algebraic properties of $M := \mathrm{coker}(\mathrm{P^t})$

$\longrightarrow$ compatibility conditions of the dirac system = syzygies of M

More in general we are interested in the free resolution:

$$0 \longrightarrow R^{\beta_s} \longrightarrow R^{\beta_{s-1}} \longrightarrow \ldots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow M \longrightarrow 0$$

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More in general we are interested in the free resolution:

$$0 \longrightarrow R^{2^n \beta'_s} \longrightarrow R^{2^n \beta'_{s-1}} \longrightarrow \ldots \longrightarrow R^{2^n \beta'_1} \longrightarrow R^{2^n \beta'_0} \longrightarrow M \longrightarrow 0$$

ALTERNATIVE

Forms on $\mathbb{R}^m$: $F_0 = C^\infty(\mathbb{R}^k)$, $F_1 = C^\infty(\mathbb{R}^k) \otimes \Lambda^1(\mathbb{R}^k)$ ...

a 1-**form** is $g_1 dx_1 + \cdots + g_k dx_k$

differential $d = \frac{\partial}{\partial x_1} dx_1 + \cdots + \frac{\partial}{\partial x_k} dx_k$

this construction leads to the **De Rham** complex :

$$0 \longrightarrow F_0 \xrightarrow{d} F_1 \xrightarrow{d} F_2 \ldots \xrightarrow{d} F_{k-1} \xrightarrow{d} F_k \longrightarrow 0$$

IDEA

Use the **radial relations** for Dirac derivatives:

$$[\{\partial_{\underline{x}_i}, \partial_{\underline{x}_j}\}, \partial_{\underline{x}_\ell}] = 0.$$

Radial algebra: associative algebra $\mathcal{R} = \langle \partial_{\underline{x}_1}, \ldots, \partial_{\underline{x}_k} \rangle$ satisfying the radial relations, e.g.

$$\partial_{\underline{x}_1}\partial_{\underline{x}_2}\partial_{\underline{x}_3} = \partial_{\underline{x}_3}\partial_{\underline{x}_2}\partial_{\underline{x}_1} + \partial_{\underline{x}_3}\partial_{\underline{x}_1}\partial_{\underline{x}_2} - \partial_{\underline{x}_2}\partial_{\underline{x}_1}\partial_{\underline{x}_3}.$$

$$\partial^2_{\underline{x}_1}\partial_{\underline{x}_2} = \partial_{\underline{x}_2}\partial^2_{\underline{x}_1}.$$

IDEA

Use the **radial relations** on Dirac operators:

$$[\{\partial_i, \partial_j\}, \partial_\ell] = 0.$$

Radial algebra: associative algebra $\mathcal{R} = \langle \partial_1, \ldots, \partial_k \rangle$

satisfying the radial relations, e.g.

$$\partial_1 \partial_2 \partial_3 = \partial_3 \partial_2 \partial_1 + \partial_3 \partial_1 \partial_2 - \partial_2 \partial_1 \partial_3.$$

$$\partial_1^2 \partial_2 = \partial_2 \partial_1^2.$$

MEGAFORMS

$k = 3,\ n > 5$

linear megaforms $D_1^i, D_2^i, D_3^i, \quad i \in \mathbb{N}$

quadratic megaforms: $D_{11}^i, D_{12}^i, D_{21}^i, D_{13}^i \ldots \quad i \in \mathbb{N}$

$$F_0 = C^\infty((\mathbb{R}^n)^3, \mathcal{C}_n)$$

$$d^0 = D_1^0 \partial_1 + D_2^0 \partial_2 + D_3^0 \partial_3 \ : F_0 \longrightarrow F_1$$

F_1: $D_1^0 g_1 + D_2^0 g_2 + D_3^0 g_3, \quad g_i \in F_0$

$$d^1 = D_1^1 \partial_1 + D_2^1 \partial_2 + D_3^1 \partial_3 + D_{11}^1 \partial_1 \partial_1 + D_{12}^1 \partial_1 \partial_2 + D_{21}^1 \partial_2 \partial_1 + \cdots + \cdots$$

$$F_2 = D_1^1 h_1 + D_2^1 h_2 + D_3^1 h_3 \ + \ D_{11}^1 h_{11} + D_{12}^1 h_{12} + D_{21}^1 h_{21} + \cdots + D_{33}^1 h_{33}$$

ALGORITHM

- *reduce* $d^1 d^0 = 0$ using radial relations $\longrightarrow$ find **relations on megaforms**

- *reduce* $d^1 g_1 = 0$ using the relations on megaforms $\longrightarrow$ find **first syzygies**, define $g_2 = d^1 g_1$

...

- *reduce* $d^N d^{N-1} = 0$ using radial relations $\longrightarrow$ find (further) **relations on megaforms**

- *reduce* $d^N g_N = 0$ using the relations on megaforms $\longrightarrow$ find N**th syzygies**

$$0 \longrightarrow F_0 \xrightarrow{d^0} F_1 \xrightarrow{d^1} F_2 \ldots \xrightarrow{d^2} F_{N-1} \xrightarrow{d^{N-1}} F_N \longrightarrow \ldots$$

EXAMPLE k=2

$$d^1 d^0 = D_{11}D_1\partial_1^3 + D_{12}D_1\partial_1\partial_2\partial_1 + D_{21}D_1\partial_2\partial_1^2 + D_{22}D_1\partial_2^2\partial_1 +$$

$$+D_{11}D_2\partial_1^2\partial_2 + D_{12}D_2\partial_1\partial_2^2 + D_{21}D_2\partial_2\partial_1\partial_2 + D_{22}D_2\partial_2^3 = 0$$

$$\downarrow$$

$$(D_{21}D_1 + D_{11}D_2)\partial_2\partial_1^2 + (D_{22}D_1 + D_{12}D_2)\partial_1\partial_2^2 +$$

$$+D_{11}D_1\partial_1^3 + D_{12}D_1\partial_1\partial_2\partial_1 + D_{21}D_2\partial_2\partial_1\partial_2 + D_{22}D_2\partial_2^3 = 0$$

$$\downarrow$$

$$D_{11}D_1 = D_{12}D_1 = D_{21}D_2 = D_{22}D_2 = 0$$

$$D_{11}D_2 + D_{21}D_1 = D_{22}D_1 + D_{12}D_2 = 0$$

EXAMPLE k=2 (cont)

$$d^1g = D_{11}D_1\partial_1^2 g_1 + D_{11}D_2\partial_1^2 g_2 + D_{12}D_1\partial_1\partial_2 g_1 + D_{12}D_2\partial_1\partial_2 g_2$$

$$+D_{21}D_1\partial_2\partial_1 g_1 + D_{21}D_2\partial_2\partial_1 g_2 + D_{22}D_1\partial_2^2 g_1 + D_{22}D_2\partial_2^2 g_2 = 0.$$

$$\downarrow$$

$$D_{21}D_1(\partial_1^2 g_2 - \partial_2\partial_1 g_1) + D_{12}D_2(\partial_2^2 g_1 - \partial_1\partial_2 g_2) = 0$$

compatibility conditions: $\partial_1^2 g_2 - \partial_2\partial_1 g_1 = 0$, $\partial_2^2 g_1 - \partial_1\partial_2 g_2$

CoCoA

GOOD: $D_1^3\, D_3^2\, D_{23}^1\, D_3^0\, \partial_1\, \partial_3\, \partial_2\, \partial_3^2$

BAD: $D_1^2\, D_3^3\, D_{23}^1\, D_3^0\, \partial_1\, \partial_3\, \partial_2\, \partial_3^2$

$\longrightarrow$ relations on megaforms are **squarefree** polynomials in

$$\mathbb{Z}_3[D_j^i, D_{k\ell}^i]$$

with decreasing superscripts!

CoCoA

GOOD: $D_1^3\, D_3^2\, D_{23}^1\, D_3^0\, \partial_1\, \partial_3\, \partial_2\, \partial_3^2$

$$\downarrow$$

`[l[3,1]l[2,3]q[1,2,3]l[0,3],[1,3,2,3,3]]`

`l[3,1]l[2,3]q[1,2,3]l[0,3]` can be reduced with Gröbner Basis

`[1,3,2,3,3]` is reduced manipulating the list looking for squares and triples (i, j, ℓ) where $i < j$ and $i < \ell$ to a "normal form"

$$\partial_{i_1} \cdots \partial_{i_s} \cdot \partial_{j_1}^2 \cdots \partial_{j_t}^2$$

$$i_a \neq \min(i_a, i_{a+1}, i_{a+2}),\ a = 1 \ldots s-2, \quad \text{and} \quad j_1 \leq \cdots \leq j_t$$

RESULTS

$M := F_0 = C^\infty(\mathbb{R}^n)^k, \mathcal{C}_n)$

$k = 2$

megaforms: $0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow M^2 \longrightarrow M^1 \longrightarrow 0$

minimal: $0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^2 \longrightarrow M^1 \longrightarrow 0$

$k = 3$

megaforms

$$0 \longrightarrow M^1 \longrightarrow M^3 \longrightarrow M^8 \longrightarrow M^{16} \longrightarrow M^{21} \longrightarrow M^{15} \longrightarrow M^3 \longrightarrow 0$$

minimal:

$$0 \longrightarrow M^1 \longrightarrow M^3 \longrightarrow M^8 \longrightarrow M^{12} \longrightarrow M^8 \longrightarrow M^3 \longrightarrow M^1 \longrightarrow 0$$

RESULTS

$$M := F_0 = C^\infty(\mathbb{R}^n)^k, \mathcal{C}_n)$$

$k = 4$

megaforms: $0 \longrightarrow M^1 \longrightarrow M^4 \longrightarrow M^{20} \longrightarrow M^{94} \longrightarrow ??$

minimal: $0 \longrightarrow M^1 \longrightarrow M^4 \longrightarrow M^{20} \longrightarrow M^{56} \longrightarrow ??$

CONCLUSIONS

1. Optimization of radial algebra reductions (`C++` ??)

2. Complexity is still exponential...

3. Gröbner Bases for megaforms are **HUGE** ($\approx 10^2$)

4. Resolutions obtained are **not minimal**

5. Minimalization of syzygies at each step is **hard to code** due to non-commutativity

6. We reached the **same point** of the resolution for $k = 4$!