Quaternionic hyperfunctions on 5-dimensional varieties in $\mathbb{H}^2$

Fabrizio Colombo
Dipartimento di Matematica
Politecnico di Milano
Via Bonardi, 9
20133 Milano, Italy
fabrizio.colombo@polimi.it

Alberto Damiano
Mathematics Department (MUUK)
Charles University
Sokolovská 83
Prague 1, 110 00 Czech Republic
damiano@karlin.mff.cuni.cz

Irene Sabadini
Dipartimento di Matematica
Politecnico di Milano
Via Bonardi, 9
20133 Milano, Italy
irene.sabadini@polimi.it

Daniele C. Struppa
Department of Mathematics and Computer Science
Chapman University
One University Drive
Orange, CA 92867, USA
struppa@chapman.edu

Abstract

In this paper we introduce a new notion of differential forms to describe the cohomology associated to the sheaf of regular functions in several quaternionic variables. We then use these differential forms to introduce and describe concretely a sheaf of quaternionic hyperfunctions as boundary values of regular functions in two quaternionic variables. We show how these ideas can be generalized to the case of monogenic functions in two vector variables with values in a Clifford algebra.

Key words: Clifford algebras, quaternions, regular and monogenic functions, differential forms, hyperfunctions.
Mathematical Review Classification numbers: 30G35, 46F15, 13D25.

1 Introduction

The definition of hyperfunctions for one real variable is relatively simple, as the space of hyperfunctions can be represented as the quotient of two spaces of holomorphic functions. Specifically, if Ω is an open set in $\mathbb{R}$, then the space of hyperfunctions $\mathcal{B}(\Omega)$ on Ω, is defined as the quotient

$$\mathcal{O}(\mathbb{C})/\mathcal{O}(\mathbb{C} \setminus \Omega),$$

where $\mathcal{O}$ denotes the sheaf of holomorphic functions. When we consider the case of several variables, however, the definition itself becomes more complicated since hyperfunctions on $\mathbb{R}^n$ are defined as n-dimensional relative cohomology classes for the sheaf of holomorphic functions. The reason why this works is hidden in the cohomological properties of the sheaf $\mathcal{O}$ and in particular the fact that the embedding of $\mathbb{R}^n$ in $\mathbb{C}^n$ is purely n-codimensional with respect to the sheaf $\mathcal{O}$. We refer the reader to [11] or to [12] for more details on this aspect.

Given the many similarities between the theory of regular functions of quaternionic variables and the classical theory of holomorphic functions, it is no surprise that several attempts have been made towards the creation of a theory of quaternionic hyperfunctions to be interpreted as suitable boundary values of regular functions. The state of the art on this problem is given in the recent monograph [6] (but see also [17] and references therein for a discussion of a different theory of hyperfunctions in one variable and values in a Clifford algebra).

All the existing work, however, only deals with rather simple extensions of the theory of one-dimensional hyperfunctions. In other words, the existing literature always consider solutions to a single differential equation (though of course both the Cauchy-Fueter and the Dirac equation can be thought of as systems). Because of the way in which the Cauchy-Fueter and the Dirac operators are defined, the theory of regular functions in one quaternionic variable and the theory of monogenic functions are essentially uni-dimensional theories. As a consequence, the corresponding hyperfunction and microfunction theories always share the flavor of the standard theory of hyperfunctions in one variable.

One of the motivations for the works which are the basis for the monograph [6] is the attempt to find a way to develop a hyperfunction theory for boundary values of regular and monogenic functions in several quaternionic or vector variables. Despite some partial progress, very little has been done in the last ten years in this direction.

This paper introduces a completely new way to address this problem, and provides a non-trivial way to build a theory of hyperfunctions for suitable cohomology classes of solutions of systems of two Cauchy-Fueter (or two Dirac) equations. The techniques which we use seem to rely essentially on the fact that we are only considering two equations, though we believe it should be possible to extend our results to higher dimensional cases.

We now introduce the notation and we sketch the strategy of this paper. We will confine ourself to the quaternionic case, since the ideas are translated to the case of Dirac systems with a few modifications.

We denote by $\mathbb{H}$ the algebra of real quaternions and by $q = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ a quaternion, where $x_\ell \in \mathbb{R}$ for $\ell = 0, ..., 3$. The symbol $\mathbb{H}_\mathbb{C}$ will denote the algebra of complex quaternions.

We define the Cauchy-Fueter operator as

$$\frac{\partial}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2} + \mathbf{k}\frac{\partial}{\partial x_3}$$

and the conjugate Cauchy-Fueter operator as

$$\frac{\partial}{\partial q} = \frac{\partial}{\partial x_0} - \mathbf{i}\frac{\partial}{\partial x_1} - \mathbf{j}\frac{\partial}{\partial x_2} - \mathbf{k}\frac{\partial}{\partial x_3}$$

with obvious meaning of the symbols. Differentiable functions which belong to the kernel of $\partial/\partial\bar{q}$ (resp. $\partial/\partial q$) are called regular (resp. anti-regular) functions.

The theory of these functions is very well established, and in fact it resembles very closely the theory of holomorphic functions in one variable (the similarity is not simply formal, but in fact the two theories share profoundly similar structures; we refer the reader to [14] for an in depth analysis of the relationship between these theories). For this reason, it is not too difficult to construct a theory for quaternionic hyperfunctions as boundary values of regular functions. This theory, which is described for example in [6] (but see also the specific references therein), leads to a flabby sheaf on $\{q \in \mathbb{H} : x_0 = 0\}$. This sheaf behaves naturally like the sheaf of classical hyperfunctions on $\mathbb{R}$, and the theory does not yield any surprise. The situation becomes rapidly more complex when we consider more than one quaternionic variable. In this paper we will concentrate our attention to the case of two quaternionic variables which, as shown in [6], is peculiar and needs to be treated separately with respect to the other cases.

Let us now consider the sheaf of regular functions in n quaternionic variables. This sheaf, as shown for example in [6], satisfies many of the same properties which are known for the sheaf of holomorphic functions. Much in the same way in which one has the Malgrange theorem, which proves the vanishing of the holomorphic cohomology of order at least n of any open set in $\mathbb{C}^n$, we can prove that the cohomology of any open set in $\mathbb{H}^n$ with coefficients in the sheaf of regular functions vanishes if the order is at least $2n-1$. The reason for this particular index is described in detail in [6] but it is essentially connected with the dimension of the characteristic variety of the Cauchy-Fueter system in n variables. This suggests that one should be able to define quaternionic hyperfunctions on $(2n+1)$-dimensional real varieties in $\mathbb{H}^n$, and that one could prove the appropriate pure codimensionality.

Unfortunately, this scheme quickly breaks down, as the fundamental instruments which allow us to perform some of the cohomological computations in the holomorphic case are a consequence of the existence of the Dolbeault sequence. Nothing of this sort exists in the quaternionic case (some sequences have been described in [4] as well as in [5] and [7], but none of them actually satisfies our necessities), and so until now there has been no theory for multivariable quaternionic hyperfunctions.

In a recent paper, [15], we have however introduced a notion of differential forms (we call them megaforms) which happen to be the missing ingredient in this strategy. The megaforms which we have introduced in [15] are actually defined for the study of systems of several Dirac operators, but in this paper we will show the kind of modifications which are necessary to treat the quaternionic case. It is interesting to note that while we had originally envisioned these objects as a computational support for the actual computation of the syzygies of these systems, they can also be used, in analogy to what happens in the holomorphic case, to provide an explicit representation of quaternionic hyperfunctions.

The generalization of our ideas to the case of two Dirac operators is not difficult, and is the object of one of our sections. We have been unable, so far, to extend the theory of megaforms to the case of three Cauchy-Fueter operators.

And now a couple of words on the plan of the paper. In section 2, we discuss the Cauchy problem for Cauchy-Fueter systems. Our major tool here is the theory developed by Ehrenpreis and Palamodov, on the general structure of the Cauchy problem. Section 3 is the most computationally intensive part of the paper. In this section we fully develop the theory of quaternionic megaforms for the case of two variables. The computations are somewhat lengthy, and we have posted some of them on our web page [8], in order to save some space, and yet enable the reader to verify the computations. Section 4 employs the results of sections two and three to finally define and study hyperfunctions for two quaternionic variables. These hyperfunctions turn out to be defined on five-dimensional subvarieties of $\mathbb{H}^2$, and can be identified with suitable cohomology classes of megaforms. In section 5 we discuss some of the modifications which are necessary to extend our theory to the case of Dirac operators.

Aknowledgements. The authors are grateful to F. Sommen for suggesting the notion of megaforms as an alternative way to describe the syzygies of the abstract Dirac system. The authors also wish to thank J. Ryan for the helpful conversations and George Mason University for partially supporting this work.

2 The Cauchy problem for the Cauchy-Fueter systems

In this section we follow the general theory of Ehrenpreis and Palamodov to understand the nature of the initial varieties for the Cauchy-Fueter system. The reason why we are interested in initial varieties for the Cauchy-Fueter system is their importance in the construction of a theory of hyperfunctions.

We begin by recalling some definitions and results due to Ehrenpreis and Palamodov ([9] and [13]). First we point out that much information on any system of linear constant coefficients differential operators is contained in what is called its characteristic variety (this is the standard terminology though it can engender some confusion when one thinks of characteristic directions for an operator). In this case, the characteristic variety V of the Cauchy-Fueter system in n variables has dimension $2n+1$ (see [3] or Chapter 3 in [6]) and it consists of points $(Q_1, \dots, Q_n)$ in $\mathbb{H}_{\mathbb{C}}^n$ such that the rank of the matrix associated to the system is not maximum. This fact, which has not been discussed in our earlier works, is important because we know (see e.g. [9], chapter 9) that the dimension of the initial varieties for the Cauchy problem coincides (at least for systems with good cohomological properties) with the dimension of its characteristic variety. Though all the results of this section will hold for arbitrary n, the rest of the paper relies on the restriction $n = 2$ and so we will simplify our notation by limiting our statements to that particular case as well. When $n = 2$ the explicit description for V is given as follows: set $Q_1 = z_0 + \mathbf{i}z_1 + \mathbf{j}z_2 + \mathbf{k}z_3$, $Q_2 = w_0 + \mathbf{i}w_1 + \mathbf{j}w_2 + \mathbf{k}w_3$ with z_ℓ, $w_\ell \in \mathbb{C}$ then

$$V = \{\ Q_1, Q_2 \in \mathbb{H}_{\mathbb{C}} \quad \text{such that} \quad z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0 \text{ and } Q_2 = QQ_1 \text{ for some } Q \in \mathbb{H}_{\mathbb{C}}\ \}.$$

Let L be an arbitrary linear subspace in $\mathbb{R}^8$ having codimension 3. We want to study the Cauchy problem for the Cauchy-Fueter system in two quaternionic variables with initial conditions given on L. In the general theory the most convenient choice is a system of coordinates in which the subspace L coincides with the coordinates subspace associated to $5 = \dim(L)$ variables.

Let us now consider the Cauchy problem for the Cauchy-Fueter system, i.e.

$$\begin{cases} \dfrac{\partial}{\partial \bar{q}_1} f = 0 \\[2ex] \dfrac{\partial}{\partial \bar{q}_2} f = 0 \\[2ex] f|_L = g \end{cases} \tag{1}$$

where g is a real analytic function in the chosen five variables (note that we cannot ask anything less on the function g since the restriction of a regular function to a real analytic variety is always real analytic). We are looking in this case for solutions in the space of $\mathcal{C}^\infty$ functions, although the ellipticity of the Cauchy-Fueter system makes this request redundant.

Remark 2.1. In the introduction the Cauchy-Fueter operator has been written using x_ℓ variables. In this section, since we have to distinguish between the original and the dual variables, we are denoting the original variables with Greek letters ζ_ℓ and the dual variables with Roman letters z_ℓ. The two Cauchy-Fueter operators will be written as

$$\frac{\partial}{\partial \bar{q}_1} = \frac{\partial}{\partial \zeta_1} + \mathbf{i}\frac{\partial}{\partial \zeta_2} + \mathbf{j}\frac{\partial}{\partial \zeta_3} + \mathbf{k}\frac{\partial}{\partial \zeta_4}, \qquad \frac{\partial}{\partial \bar{q}_2} = \frac{\partial}{\partial \zeta_5} + \mathbf{i}\frac{\partial}{\partial \zeta_6} + \mathbf{j}\frac{\partial}{\partial \zeta_7} + \mathbf{k}\frac{\partial}{\partial \zeta_8}.$$

In general the Cauchy problem is not well posed unless we make suitable assumptions on L. To this aim we state the result of Ehrenpreis and Palamodov as follows (see page 226 in [9] and page 284 in [13]), but first we need some notation. We define the following splitting of the variables in $\mathbb{R}^s$: suppose that L corresponds to the variables $\zeta'' = (\zeta_{m+1}, ..., \zeta_s)$ and set $\zeta' = (\zeta_1, ..., \zeta_m)$. We will denote by $\mathbb{R}_{\zeta'}$ and $\mathbb{R}_{\zeta''}$ the corresponding coordinate subspaces in $\mathbb{R}^s$. Accordingly, in the space of the dual variables we have the splitting $z' = (z_1, ..., z_m)$ and $z'' = (z_{m+1}, ..., z_s)$, $z_\ell \in \mathbb{C}$. If $z_\ell = x_\ell + iy_\ell$, we denote by $x' = (x_1, ..., x_m)$, $y' = (y_1, ..., y_m)$ the vectors corresponding to the real and the imaginary part of z'. So we have:

Theorem 2.2. *Let P be a rectangular matrix with entries in the ring of polynomials $\mathbb{C}[z_1, \dots, z_s]$ and let $D = (-i\frac{\partial}{\partial \zeta_1}, \dots, -i\frac{\partial}{\partial \zeta_s})$. Let V be the characteristic variety associated to the system represented by the operator $P(D)$, i.e., the set of points in $\mathbb{C}^s$ where P is not maximal rank. Suppose that on the variety V the operator $P(D)$ is weakly hypoelliptic in the variable ζ', i.e., the inequality*

$$|z'| \leq B(|y'| + |z''| + 1)^{1/\gamma}, \quad B > 0, \quad \gamma > 0$$

holds on the variety V. Then the Cauchy problem associated to $P(D)f = 0$ with initial condition given on L has a unique solution in the space of distributions defined on the region $\Omega = \Omega' \times \mathbb{R}_{\zeta''}$ where $\Omega' = \mathbb{R}_{\zeta'}$ or $\Omega' = \{\zeta' \in \mathbb{R}_{\zeta'} \ : \ |\zeta'| < A\}$ where A is an arbitrary positive constant.

We show that a particular choice of the subspace L for the Cauchy-Fueter system in two variables satisfies the hypothesis of the above theorem.

Theorem 2.3. *Let L be the subspace of $\mathbb{R}^8$ corresponding to the variables $\zeta'' = (\zeta_4, \zeta_5, \zeta_6, \zeta_7, \zeta_8)$ and let $\zeta' = (\zeta_1, \zeta_2, \zeta_3)$. The dual variables will be denoted by $z' = (z_0, z_1, z_2)$ and $z'' = (z_3, w_0, w_1, w_2, w_3)$. Then the Cauchy-Fueter operator in two variables is weakly hypoelliptic in ζ'.*

Proof. Taking the real part of the equality $z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0$ we obtain $x_0^2 + x_1^2 + x_2^2 + x_3^2 = y_0^2 + y_1^2 + y_2^2 + y_3^2$. Since

$$|z'|^2 = x_0^2 + x_1^2 + x_2^2 + y_0^2 + y_1^2 + y_2^2 = 2(y_0^2 + y_1^2 + y_2^2) + y_3^2 - x_3^2$$

and

$$|y'|^2 = y_0^2 + y_1^2 + y_2^2$$

we get

$$|z'|^2 \leq 2(y_0^2 + y_1^2 + y_2^2) + y_3^2 = 2|y'|^2 + y_3^2 \leq 2|y'|^2 + |z''|^2 \leq 2(|y'|^2 + |z''|^2).$$

This implies

$$|z'| \leq \sqrt{2}\sqrt{|y'|^2 + |z''|^2} \leq \sqrt{2}(|y'| + |z''|),$$

from which we get the statement. □

Corollary 2.4. *The subspace L is an initial variety for the Cauchy problem (1).*

Remark 2.5. The subspace L for which the Cauchy problem (1) is well posed is far from being unique. Other choices are obviously possible for example by taking different splittings of the variables.

Remark 2.6. It is immediate to note that this last theorem can in fact be proved for n Cauchy-Fueter operators. In that case any $(2n + 1)$-dimensional linear subspace L in $\mathbb{R}^{4n}$ parametrized by $\zeta'' = (\zeta_{2n}, \dots, \zeta_{4n})$ is an initial variety for the Cauchy-Fueter system.

Let us recall that in the classical construction of hyperfunctions as boundary values of holomorphic functions of several complex variables, the key result is Sato's theorem on the pure n-codimensionality of $\mathbb{R}^n$ in $\mathbb{C}^n$. In a few words, this means that one needs to prove that if we denote by $\mathcal{H}^j$ the derived sheaves of the sheaf $\mathcal{O}$ of holomorphic functions, then $\mathcal{H}^j_{\mathbb{R}^n}(\mathcal{O}) = 0$ for $j \neq n$. This actually translates in the vanishing of suitable relative cohomology groups. The proof of this result is quite involved, and the reader is referred to Chapter 6, Theorem 6.5.6 in [11] for details. What is important however is that the proof relies essentially on Grauert's theorem on the embeddability of a real analytic manifold in a complexified manifold of double real dimension. The reason for this embeddability is, in turn, the fact that real analytic manifolds are initial varieties for the Cauchy-Riemann system.

Our analysis in this section shows how to construct appropriate initial varieties for the Cauchy-Fueter system. By an argument which mimics the one given in the complex case, these are the varieties which appear as supports for hyperfunctions.

3 Quaternionic megaforms for the two dimensional Cauchy-Fueter system

In this section we use some of the ideas presented in [15] in the case of the Dirac systems to construct an appropriate theory of megaforms which can be used to study the Cauchy-Fueter system in two quaternionic variables.

Let n be a nonnegative integer and let B^n be the set

$$B^n = \left\{ D_i^n, \check{D}_i^n, D_{ij}^n, \check{D}_{ij}^n, \tilde{D}_{ij}^n, D^{*n}_{ij}, \quad i,j = 1,2 \right\},$$

whose elements we interpret as the analogue of the differential forms dz_i, $d\bar{z}_i$ used in the complex setting. We define the differentials

$$d^0 = \sum_{i=1}^{k} \check{D}_i^0 \bar{\partial}_i$$

$$d^n = \sum_{i=1}^{k} (\check{D}_i^n \bar{\partial}_i + D_i^n \partial_i) + \sum_{i,j=1}^{k} (\check{D}_{ij}^n \bar{\partial}_i \bar{\partial}_j + D_{ij}^n \partial_i \partial_j + \tilde{D}_{ij}^n \partial_i \bar{\partial}_j + D^{*n}_{ij} \bar{\partial}_i \partial_j),$$

where $\bar{\partial}_i$ and ∂_i denote $\partial/\partial\bar{q}_i$ and $\partial/\partial q_i$, respectively. Let $\mathcal{Q}$ be the associative algebra generated by the symbols $\bar{\partial}_i$ and ∂_i, $i = 1 \dots k$, together with the relations given by the fact that the Laplacian $\Delta_i := \partial_i \bar{\partial}_i = \bar{\partial}_i \partial_i$ commutes with every generator.

Definition 3.1. *Let $\mathcal{B}$ be the free algebra generated over $\mathbb{C}$ by $\cup_{n \geq 0} B^n$. Let $\mathcal{B} \otimes \mathcal{Q}$ the tensor product of $\mathcal{B}$ and $\mathcal{Q}$ as $\mathbb{C}$-algebras, and let $d^n \in \mathcal{B} \otimes \mathcal{Q}$, $n \in \mathbb{N}$, be as defined above. The algebra of quaternionic megaforms $\mathcal{M}$ is the quotient of $\mathcal{B} \otimes \mathcal{Q}$ with the left ideal generated by $\{d^{n+1} \cdot d^n \mid n \in \mathbb{N}\}$.*

Basically, $\mathcal{M}$ is the (non-commutative) algebra of differential forms $\mathcal{B}$ with right coefficients in $\mathcal{Q}$ in which the closure relations $d^{n+1} \cdot d^n$ hold. Note that the superscript "n" on the symbols in B^n is not strictly necessary if the non commutative symbols in B^n are considered ordered increasingly from right to left. For this reason, from now on, we will simply omit the superscripts.

Let $\mathcal{R}$ be the space of regular functions in two variables q_1, q_2. Let F_0 be the space of $\mathcal{C}^\infty$ functions in q_1, q_2 with quaternionic values. It is a left $\mathcal{Q}$-module in the obvious way. Let

$\mathcal{F} = \mathcal{M} \otimes_{\mathcal{Q}} F_0$. An element $m \otimes g$ of $\mathcal{F}$ can be written as $D \otimes g'$ with $D \in \mathcal{B}$ and $g' \in F_0$ after the coefficient of m, which is an operator in $\mathcal{Q}$, acts on the function g. This says that there is an injective map from $\mathcal{M} \otimes_{\mathcal{Q}} F_0$ to $\mathcal{B} \otimes_{\mathbb{C}} F_0$. From now on, we will then omit the tensor product notation and just use juxtaposition. Our goal is now to construct a Dolbeault-like sequence

$$0 \to \mathcal{R} \longrightarrow F_0 \xrightarrow{d^0} F_1 \xrightarrow{d^1} F_2 \xrightarrow{d^2} \dots \xrightarrow{d^k} F_{k+1} \xrightarrow{d^{k+1}} \dots \tag{2}$$

for suitable spaces F_i and where the differentials are naturally extended to such spaces. The sequence will be a complex by definition of $\mathcal{M}$. Let F_1 be the subspace of $\mathcal{F}$ of 1-megaforms, whose elements are written as

$$g = \sum_{i=1}^{2} \check{D}_i g_i, \quad g_i \in F_0, \quad i = 1, 2.$$

Then we have an exact sequence

$$0 \to \mathcal{R} \hookrightarrow F_0 \overset{d^0}{\to} F_1.$$

The next step in the construction of the complex consists in defining a space F_2 of 2-megaforms and introduce $d^1 : F_1 \to F_2$, so that $d^1 d^0 = 0$ by the defining relations of $\mathcal{M}$. We postulate (see also [15]) that d^1 be made of two components d_1^1 and d_2^1 of degrees, respectively, one and two. Let us define

$$d_1^1 = \check{D}_1 \bar{\partial}_1 + \check{D}_2 \bar{\partial}_2 + D_1 \partial_1 + D_2 \partial_2,$$

$$d_2^1 = \sum_{i,j=1}^{2} (\check{D}_{ij} \bar{\partial}_i \bar{\partial}_j + D_{ij} \partial_i \partial_j + \tilde{D}_{ij} \partial_i \bar{\partial}_j + D^*_{ij} \bar{\partial}_i \partial_j),$$

and let us set $d^1 = d_1^1 + d_2^1$.

Remark 3.2. If we impose the condition $d^1 d^0 = 0$ then, in particular, we have $d_1^1 d^0 = 0$, i.e.

$$\sum_{j,k=1}^{2} (\check{D}_k \check{D}_j \bar{\partial}_k \bar{\partial}_j + D_k \check{D}_j \partial_k \bar{\partial}_j) f = 0.$$

As a consequence, for any k and j, $\check{D}_k \check{D}_j = 0$ and $D_k \check{D}_j = 0$. So one can assume that $d_1^1 \equiv 0$, and therefore $d^1 = d_2^1$.

Proposition 3.3. *Let*

$$d^0 = \check{D}_1 \bar{\partial}_1 + \check{D}_2 \bar{\partial}_2$$

and

$$d^1 = \sum_{i,j=1}^{2} (\check{D}_{ij} \bar{\partial}_i \bar{\partial}_j + D_{ij} \partial_i \partial_j + \tilde{D}_{ij} \partial_i \bar{\partial}_j + D^*_{ij} \bar{\partial}_i \partial_j).$$

Then $d^1 d^0 = 0$ implies the following:

(1) $(\tilde{D}_{ii} + D^*_{ii}) \check{D}_i = 0, \quad i = 1, 2,$

(2) $D^*_{ji} \check{D}_i + (\tilde{D}_{ii} + D^*_{ii}) \check{D}_j = 0, \quad i, j = 1, 2, \ i \neq j,$

(3) $D_{ij} \check{D}_k = 0, \quad i, j, k = 1, 2,$

(4) $\check{D}_{ij} \check{D}_k = 0, \quad i, j, k = 1, 2,$

(5) $\tilde{D}_{ij}\check{D}_k = 0, \quad i,j,k = 1,2, \quad i \neq j,$

(6) $D^*_{ij}\check{D}_i = 0, \quad i,j = 1,2, \quad i \neq j.$

Proof. The condition $d^1_2 d^0 = 0$ implies that for any $f \in F_0$ we have

$$\sum_{i,j=1}^{2} (\check{D}_{ij}\check{D}_k\bar{\partial}_i\bar{\partial}_j\bar{\partial}_k + D_{ij}\check{D}_k\partial_i\partial_j\bar{\partial}_k + \tilde{D}_{ij}\check{D}_k\partial_i\bar{\partial}_j\bar{\partial}_k + D^*_{ij}\check{D}_k\bar{\partial}_i\partial_j\bar{\partial}_k)f = 0.$$

By writing explicitly the left hand term, by using the relations $\partial_i\bar{\partial}_i\bar{\partial}_j = \bar{\partial}_j\partial_i\bar{\partial}_i$, i.e. $\Delta_i\bar{\partial}_j = \bar{\partial}_j\Delta_i$, $i,j = 1,2$ and by grouping the various terms we get the statement. □

Remark 3.4. The relations *(1)-(6)* in Proposition 3.3 are the analogues of the complex relations $d\bar{z}_i \wedge d\bar{z}_j = -d\bar{z}_j \wedge d\bar{z}_i$, $i,j = 1,2$.

The condition of closure of a 1-megaform is related to the syzygies of the non-homogeneous Cauchy-Fueter system in two variables: a 1–megaform g is d^1–closed if and only if its components g_j satisfy the compatibility conditions of the system $d^0 f = g$.

Proposition 3.5. *Let $g = \check{D}_1 g_1 + \check{D}_2 g_2$ be an element of F_1. Then $d^1 g = 0$ if and only if*

$$\bar{\partial}_i\partial_i g_j - \bar{\partial}_j\partial_i g_i = 0, \quad i,j = 1,2, \ i \neq j, \tag{3}$$

i.e., $d^1 g = 0$ if and only if g_1 and g_2 satisfy the compatibility conditions for the solvability of the system

$$\begin{cases} \bar{\partial}_1 f = g_1 \\ \bar{\partial}_2 f = g_2. \end{cases}$$

Proof. By the definition of g and d^1 we have that $d^1 g = 0$ can be written as

$$\begin{aligned} &\check{D}_{11}\check{D}_1\bar{\partial}_1^2 g_1 + \check{D}_{12}\check{D}_1\bar{\partial}_1\bar{\partial}_2 g_1 + \check{D}_{21}\check{D}_1\bar{\partial}_2\bar{\partial}_1 g_1 + \check{D}_{22}\check{D}_1\bar{\partial}_2^2 g_1 \\ &+D_{11}\check{D}_1\partial_1^2 g_1 + D_{12}\check{D}_1\partial_1\partial_2 g_1 + D_{21}\check{D}_1\partial_2\partial_1 g_1 + D_{22}\check{D}_1\partial_2^2 g_1 \\ &+\tilde{D}_{11}\check{D}_2\partial_1^2 g_2 + \tilde{D}_{12}\check{D}_2\partial_1\partial_2 g_2 + \tilde{D}_{21}\check{D}_2\partial_2\partial_1 g_2 + \tilde{D}_{22}\check{D}_2\partial_2^2 g_2 \\ &+D_{11}\check{D}_2\partial_1^2 g_2 + D_{12}\check{D}_2\partial_1\partial_2 g_2 + D_{21}\check{D}_2\partial_2\partial_1 g_2 + D_{22}\check{D}_2\partial_2^2 g_2 = 0. \end{aligned}$$

In view of *(1)–(6)* in Proposition 3.3 this can be rewritten as

$$\begin{aligned} &D^*_{11}\check{D}_1(\bar{\partial}_1\partial_1 g_1 - \partial_1\bar{\partial}_1 g_1) + D^*_{22}\check{D}_2(\bar{\partial}_2\partial_2 g_2 - \partial_2\bar{\partial}_2 g_2) \\ &+D^*_{22}\check{D}_1(\bar{\partial}_2\partial_2 g_1 - \bar{\partial}_1\partial_2 g_2) + \tilde{D}_{22}\check{D}_1(\partial_2\bar{\partial}_2 g_1 - \bar{\partial}_1\partial_2 g_2) \\ &+D^*_{11}\check{D}_2(\bar{\partial}_1\partial_1 g_2 - \bar{\partial}_2\partial_1 g_1) + \tilde{D}_{11}\check{D}_2(\partial_1\bar{\partial}_1 g_2 - \bar{\partial}_2\partial_1 g_1) = 0, \end{aligned}$$

which completes the proof. □

From the general theory (see [6]), it is known that the complex closes with one more linear condition that is the compatibility condition for the solvability of the system

$$\begin{cases} \Delta_1 g_2 - \bar{\partial}_2\partial_1 g_1 = h_{12} \\ \Delta_2 g_1 - \bar{\partial}_1\partial_2 g_2 = h_{21}. \end{cases} \tag{4}$$

We derive this condition using megaforms and their closure.

Proposition 3.6. *Let*

$$d^1 = \sum_{i,j=1}^{2} (\check{D}_{ij}\bar{\partial}_i\bar{\partial}_j + D_{ij}\partial_i\partial_j + \tilde{D}_{ij}\partial_i\bar{\partial}_j + D^*_{ij}\bar{\partial}_i\partial_j) \tag{5}$$

and let

$$\begin{aligned} d^2 = d_1^2 + d_2^2 \ &= \check{D}_1\bar{\partial}_1 + \check{D}_2\bar{\partial}_2 + D_1\partial_1 + D_2\partial_2 \\ &+ \textstyle\sum_{i,j=1}^{2}(\check{D}_{ij}\bar{\partial}_i\bar{\partial}_j + D_{ij}\partial_i\partial_j + \tilde{D}_{ij}\partial_i\bar{\partial}_j + D^*_{ij}\bar{\partial}_i\partial_j). \end{aligned} \tag{6}$$

Then $d^2d^1 = 0$ implies, for $i, j = 1, 2,\ i \neq j$, the conditions

(1) $\check{D}_iD^*_{ii}\check{D}_j + \check{D}_i\tilde{D}_{ii}\check{D}_j = 0, \quad i, j = 1, 2, \quad i \neq j,$

(2) $\check{D}_iD^*_{jj}\check{D}_i + \check{D}_i\tilde{D}_{jj}\check{D}_i = 0, \quad i, j = 1, 2, \quad i \neq j,$

(3) $D_iD^*_{ii}\check{D}_j + D_i\tilde{D}_{ii}\check{D}_j = 0, \quad i, j = 1, 2, \quad i \neq j,$

(4) $(D_1D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1) - (D_2D^*_{11}\check{D}_2 + D_2\tilde{D}_{11}\check{D}_2) = 0,$

(5) $\tilde{D}_{ii}D^*_{ii}\check{D}_j + \tilde{D}_{ii}\tilde{D}_{ii}\check{D}_j + D^*_{ii}D^*_{ii}\check{D}_j + D^*_{ii}\tilde{D}_{ii}\check{D}_j = 0, \quad i, j = 1, 2, \quad i \neq j,$

(6) $D_{ii}D^*_{jj}\check{D}_i + D_{ii}\tilde{D}_{jj}\check{D}_i - D_{ij}D^*_{ii}\check{D}_j - D_{ij}\tilde{D}_{ii}\check{D}_j = 0, \quad i, j = 1, 2, \quad i \neq j,$

(7) $\tilde{D}_{ii}D^*_{jj}\check{D}_i + \tilde{D}_{ii}\tilde{D}_{jj}\check{D}_i + D^*_{ii}D^*_{jj}\check{D}_i + D^*_{ii}\tilde{D}_{jj}\check{D}_i$
$- D^*_{ij}D^*_{ii}\check{D}_j - D^*_{ij}\tilde{D}_{ii}\check{D}_j = 0, \quad i, j = 1, 2, \quad i \neq j,$

(8) $\check{D}_{ij}D^*_{kk}\check{D}_\ell + \check{D}_{ij}\tilde{D}_{kk}\check{D}_\ell = 0, \quad i, j, k, \ell = 1, 2, \quad k \neq \ell,$

(9) $D_{ii}D^*_{ii}\check{D}_j + D_{ii}\tilde{D}_{ii}\check{D}_j = 0, \quad i, j, = 1, 2, \quad i \neq j,$

(10) $\tilde{D}_{ij}D^*_{kk}\check{D}_\ell + \tilde{D}_{ij}\tilde{D}_{kk}\check{D}_\ell = 0, \quad i, j, k, \ell = 1, 2, \quad k \neq \ell, \quad i \neq j,$

(11) $D^*_{ji}D^*_{ii}\check{D}_j + D^*_{ji}\tilde{D}_{ii}\check{D}_j = 0, \quad i, j, = 1, 2, \quad i \neq j,$

(12) $D_{ji}D^*_{ii}\check{D}_j + D_{ji}\tilde{D}_{ii}\check{D}_j = 0, \quad i, j, = 1, 2, \quad i \neq j.$

Proof. Let us consider $g \in F_1$. By the previous computations we have that d^1g is of the form

$$d^1g = (D^*_{11}\check{D}_2 + \tilde{D}_{11}\check{D}_2)(\Delta_1g_2 - \bar{\partial}_2\partial_1g_1) + (D^*_{22}\check{D}_1 + \tilde{D}_{22}\check{D}_1)(\Delta_2g_1 - \bar{\partial}_1\partial_2g_2)$$

and

$$d^2d^1g = d_1^2d^1g + d_2^2d^1g$$

so

$$\begin{aligned} d_1^2d^1g \ &= [\check{D}_1\bar{\partial}_1 + \check{D}_2\bar{\partial}_2 + D_1\partial_1 + D_2\partial_2][(D^*_{11}\check{D}_2 + \tilde{D}_{11}\check{D}_2)(\Delta_1g_2 - \bar{\partial}_2\partial_1g_1) \\ &+(D^*_{22}\check{D}_1 + \tilde{D}_{22}\check{D}_1)(\Delta_2g_1 - \bar{\partial}_1\partial_2g_2)] \\ &= (\check{D}_1D^*_{11}\check{D}_2 + \check{D}_1\tilde{D}_{11}\check{D}_2)(\bar{\partial}_1\Delta_1g_2 - \bar{\partial}_1\bar{\partial}_2\partial_1g_1) \\ &+(\check{D}_1D^*_{22}\check{D}_1 + \check{D}_1\tilde{D}_{22}\check{D}_1)(\bar{\partial}_1\Delta_2g_1 - \bar{\partial}_1\bar{\partial}_1\partial_2g_2) \\ &+(\check{D}_2D^*_{11}\check{D}_2 + \check{D}_2\tilde{D}_{11}\check{D}_2)(\bar{\partial}_2\Delta_1g_2 - \bar{\partial}_2\bar{\partial}_2\partial_1g_1) \\ &+(\check{D}_2D^*_{22}\check{D}_1 + \check{D}_2\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\Delta_2g_1 - \bar{\partial}_2\bar{\partial}_1\partial_2g_2) \\ &+(D_1D^*_{11}\check{D}_2 + D_1\tilde{D}_{11}\check{D}_2)(\partial_1\Delta_1g_2 - \partial_1\bar{\partial}_2\partial_1g_1) \\ &+(D_1D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)(\partial_1\Delta_2g_1 - \Delta_1\partial_2g_2) \\ &+(D_2D^*_{11}\check{D}_2 + D_2\tilde{D}_{11}\check{D}_2)(\partial_2\Delta_1g_2 - \Delta_2\partial_1g_1) \\ &+(D_2D^*_{22}\check{D}_1 + D_2\tilde{D}_{22}\check{D}_1)(\partial_2\Delta_2g_1 - \partial_2\bar{\partial}_1\partial_2g_2) = 0. \end{aligned} \tag{7}$$

Grouping the various terms we obtain conditions *(1)–(4)* (the detailed computations are available in [8]). Analogously, by imposing $d_2^2d^1g = 0$, we obtain (see again [8]) *(5)–(12)*. □

Remark 3.7. Using the relations *(1)–(6)* in Proposition 3.3, the elements belonging to F_2 can be written using the terms $D^*_{11}\check{D}_2 + \tilde{D}_{11}\check{D}_2$ and $D^*_{22}\check{D}_1 + \tilde{D}_{22}\check{D}_1$.

Proposition 3.8. *Let*

$$h = (D^*_{11}\check{D}_2 + \tilde{D}_{11}\check{D}_2)h_{12} + (D^*_{22}\check{D}_1 + \tilde{D}_{22}\check{D}_1)h_{21} \tag{8}$$

be a generic element in F_2. Then $d^2h = 0$ if and only if

$$\partial_1 h_{21} + \partial_2 h_{12} = 0,$$

i.e., $d^2h = 0$ if and only if $h = (h_{12}, h_{21})$ satisfies the compatibility condition for the system (4).

Proof. The condition $d^2h = 0$ can be split into $d^2_1 h = 0$ and $d^2_2 h = 0$. The first condition gives

$$\begin{aligned} d^2_1 h \; &= [\check{D}_1\bar{\partial}_1 + \check{D}_2\bar{\partial}_2 + D_1\partial_1 + D_2\partial_2][(D^*_{11}\check{D}_2 + \tilde{D}_{11}\check{D}_2)h_{12} + (D^*_{22}\check{D}_1 + \tilde{D}_{22}\check{D}_1)h_{21}] \\ &= (\check{D}_1 D^*_{11}\check{D}_2 + \check{D}_1\tilde{D}_{11}\check{D}_2)\bar{\partial}_1 h_{12} + (\check{D}_1 D^*_{22}\check{D}_1 + \check{D}_1\tilde{D}_{22}\check{D}_1)\bar{\partial}_1 h_{21} \\ &+(\check{D}_2 D^*_{11}\check{D}_2 + \check{D}_2\tilde{D}_{11}\check{D}_2)\bar{\partial}_2 h_{12} + (\check{D}_2 D^*_{22}\check{D}_1 + \check{D}_2\tilde{D}_{22}\check{D}_1)\bar{\partial}_2 h_{21} \\ &+(D_1 D^*_{11}\check{D}_2 + D_1\tilde{D}_{11}\check{D}_2)\partial_1 h_{12} + (D_1 D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)\partial_1 h_{21} \\ &+(D_2 D^*_{11}\check{D}_2 + D_2\tilde{D}_{11}\check{D}_2)\partial_2 h_{12} + (D_2 D^*_{22}\check{D}_1 + D_2\tilde{D}_{22}\check{D}_1)\partial_2 h_{21} = 0. \end{aligned} \tag{9}$$

Using the relations *(1)-(4)* given in Proposition 3.6 we obtain:

$$\partial_1 h_{21} + \partial_2 h_{12} = 0. \tag{10}$$

The condition $d^2_2 h = 0$ gives

$$\begin{aligned} d^2_2 h \; &= (D_{11} D^*_{22}\check{D}_1 + D_{11}\tilde{D}_{22}\check{D}_1)\partial^2_1 h_{21} \\ &+(\tilde{D}_{11} D^*_{11}\check{D}_2 + \tilde{D}_{11}\tilde{D}_{11}\check{D}_2)\Delta_1 h_{12} + (\tilde{D}_{11} D^*_{22}\check{D}_1 + \tilde{D}_{11}\tilde{D}_{22}\check{D}_1)\Delta_1 h_{21} \\ &+(D^*_{11} D^*_{11}\check{D}_2 + D^*_{11}\tilde{D}_{11}\check{D}_2)\Delta_1 h_{12} + (D^*_{11} D^*_{22}\check{D}_1 + D^*_{11}\tilde{D}_{22}\check{D}_1)\Delta_1 h_{21} \\ &+(D_{12} D^*_{11}\check{D}_2 + D_{12}\tilde{D}_{11}\check{D}_2)\partial_1\partial_2 h_{12} + (D^*_{12} D^*_{11}\check{D}_2 + D^*_{12}\tilde{D}_{11}\check{D}_2)\bar{\partial}_1\partial_2 h_{12} \\ &+(D_{21} D^*_{22}\check{D}_1 + D_{21}\tilde{D}_{22}\check{D}_1)\partial_2\partial_1 h_{21} + (D^*_{21} D^*_{22}\check{D}_1 + D^*_{21}\tilde{D}_{22}\check{D}_1)\bar{\partial}_2\partial_1 h_{21} \\ &+(D_{22} D^*_{11}\check{D}_2 + D_{22}\tilde{D}_{11}\check{D}_2)\partial^2_2 h_{12} + (\tilde{D}_{22} D^*_{11}\check{D}_2 + \tilde{D}_{22}\tilde{D}_{11}\check{D}_2)\Delta_2 h_{12} \\ &+(\tilde{D}_{22} D^*_{22}\check{D}_1 + \tilde{D}_{22}\tilde{D}_{22}\check{D}_1)\Delta_2 h_{21} + (D^*_{22} D^*_{11}\check{D}_2 + D^*_{22}\tilde{D}_{11}\check{D}_2)\Delta_2 h_{12} \\ &+(D^*_{22} D^*_{22}\check{D}_1 + D^*_{22}\tilde{D}_{22}\check{D}_1)\Delta_2 h_{21} = 0. \end{aligned} \tag{11}$$

Using the relations *(5)-(12)* given in Proposition 3.6, we obtain the following second order relations:

$$\begin{aligned} \partial_1(\partial_1 h_{21} + \partial_2 h_{12}) = 0, \\ \partial_2(\partial_1 h_{21} + \partial_2 h_{12}) = 0, \\ \bar{\partial}_1(\partial_1 h_{21} + \partial_2 h_{12}) = 0, \\ \bar{\partial}_2(\partial_1 h_{21} + \partial_2 h_{12}) = 0, \end{aligned} \tag{12}$$

which are implied by the linear relations already obtained. □

We now expect that there will be no more syzygies among the data of the inhomogeneous system associated to the relations obtained in Proposition 3.8. To this aim we point out that, even though (10) and (12) form a system consisting of five equations, only (10) is necessary since the other ones are consequence of (10). So the system we have to consider reduces to only one equation:

$$\partial_1 h_{21} + \partial_2 h_{12} = k_0. \tag{13}$$

Condition (10) comes from $d^2_1 d^1 = 0$ while (12) comes from $d^2_2 d^1 = 0$. Since this last part is redundant in the next step we will consider only the relations coming from $d^3 d^2_1 = 0$.

Proposition 3.9. *Let*

$$d^3 = d^3_1 + d^3_2 = \begin{array}{l} \check{D}_1\bar\partial_1 + \check{D}_2\bar\partial_2 + D_1\partial_1 + D_2\partial_2 \\ + \sum_{i,j=1}^2 (\check{D}_{ij}\bar\partial_i\bar\partial_j + D_{ij}\partial_i\partial_j + \tilde{D}_{ij}\partial_i\bar\partial_j + D^*_{ij}\bar\partial_i\partial_j). \end{array} \tag{14}$$

Then $d^3d^2_1 = 0$ *implies*

(1) $\check{D}_iD_1D^*_{22}\check{D}_1 + \check{D}_iD_1\tilde{D}_{22}\check{D}_1 = 0, \quad i = 1, 2,$

(2) $D_iD_1D^*_{22}\check{D}_1 + D_iD_1\tilde{D}_{22}\check{D}_1 = 0, \quad i = 1, 2,$

(3) $(\tilde{D}_{ii}D_1D^*_{22}\check{D}_1 + \tilde{D}_{ii}D_1\tilde{D}_{22}\check{D}_1) + (D^*_{ii}D_1D^*_{22}\check{D}_1 + D^*_{ii}D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i, j = 1, 2,\ i \neq j,$

(4) $(D_{ij}D_1D^*_{22}\check{D}_1 + D_{ij}D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i, j = 1, 2,$

(5) $(\check{D}_{ij}D_1D^*_{22}\check{D}_1 + \check{D}_{ij}D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i, j = 1, 2,$

(6) $(\tilde{D}_{ij}D_1D^*_{22}\check{D}_1 + \tilde{D}_{ij}D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i, j = 1, 2,\ i \neq j,$

(7) $(D^*_{ij}D_1D^*_{22}\check{D}_1 + D^*_{ij}D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i, j = 1, 2,\ i \neq j.$

Proof. Let $h \in F_2$ as in (8). Then, using (9) and the relations of Proposition 3.6, we can write d^2_1h as

$$d^2_1h = (D_1D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)(\partial_1h_{21} + \partial_2h_{12}).$$

Let us compute separately $d^3_1d^2_1h$ and $d^3_2d^2_1h$:

$$\begin{array}{rl} d^3_1d^2_1h & = [\check{D}_1\bar\partial_1 + \check{D}_2\bar\partial_2 + D_1\partial_1 + D_2\partial_2][(D_1D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)(\partial_1h_{21} + \partial_2h_{12})] \\ & = (\check{D}_1D_1D^*_{22}\check{D}_1 + \check{D}_1D_1\tilde{D}_{22}\check{D}_1)(\Delta_1h_{21} + \bar\partial_1\partial_2h_{12}) \\ & +(\check{D}_2D_1D^*_{22}\check{D}_1 + \check{D}_2D_1\tilde{D}_{22}\check{D}_1)(\bar\partial_2\partial_1h_{21} + \Delta_2h_{12}) \\ & +(D_1D_1D^*_{22}\check{D}_1 + D_1D_1\tilde{D}_{22}\check{D}_1)(\partial^2_1h_{21} + \partial_1\partial_2h_{12}) \\ & +(D_2D_1D^*_{22}\check{D}_1 + D_2D_1\tilde{D}_{22}\check{D}_1)(\partial_2\partial_1h_{21} + \partial^2_2h_{12}). \end{array} \tag{15}$$

Condition $d^3_1d^2_1h = 0$ implies relations *(1)* and *(2)*, so that $d^3_1d^2_1h \equiv 0$. Analogously, computing $d^3_2d^2_1h$ and setting it equal to zero, we obtain (see [8] for the detailed computations) the relations *(3)-(7)* listed in the statement. □

Proposition 3.10. *Let*

$$k = (D_1D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)k_0$$

be a generic element in F_3*. Then* $d^3k = 0$*.*

Proof. This follows immediately from the fact that d^3_1k is identically zero, and that $d^3_2k = 0$ only yields identities of the form $\Delta_ik_0 - \Delta_ik_0 = 0$. □

Remark 3.11. If we consider the system in five equations

$$\begin{cases} \partial_1h_{21} + \partial_2h_{12} = k_0 \\ \partial_1(\partial_1h_{21} + \partial_2h_{12}) = k_1 \\ \partial_2(\partial_1h_{21} + \partial_2h_{12}) = k_2 \\ \bar\partial_1(\partial_1h_{21} + \partial_2h_{12}) = k_3 \\ \bar\partial_2(\partial_1h_{21} + \partial_2h_{12}) = k_4, \end{cases}$$

instead of the single equation (13), then Proposition 3.9 needs to be rewritten taking into account all the relations coming from the vanishing $d^3d^2 = 0$. These computations, available on [8], show that the relations among the various k_i, $i = 0, \ldots, 4$, subjected to the constraints $\partial_ik_0 = k_i$, $\bar\partial_ik_0 = k_{i+2}$, $i = 1, 2$ only give identities.

We can summarize the results in this section in the following:

Theorem 3.12. *Regular functions in two quaternionic variables can be embedded in the following Dolbeault–like complex:*

$$0 \to \mathcal{R} \hookrightarrow F_0 \overset{d^0}{\to} F_1 \overset{d^1}{\to} F_2 \overset{d^2}{\to} F_3 \overset{d^3}{\to} 0, \tag{16}$$

where

$$\begin{aligned}
&F_0 := \mathcal{C}^\infty,\\
&d^0 = \check{D}_1\bar{\partial}_1 + \check{D}_2\bar{\partial}_2,\\
&\\
&F_1 := \{\textstyle\sum_{i=1}^2 \check{D}_i\check{g}_i, \quad \check{g}_i \in \mathcal{C}^\infty\},\\
&d^1 = d^1_2 = \textstyle\sum_{i,j=1}^2(\check{D}_{ij}\bar{\partial}_i\bar{\partial}_j + D_{ij}\partial_i\partial_j + \tilde{D}_{ij}\partial_i\bar{\partial}_j + D^*_{ij}\bar{\partial}_i\partial_j),\\
&\\
&F_2 := \{(D^*_{11}\check{D}_2 + \tilde{D}_{11}\check{D}_2)h_{12} + (D^*_{22}\check{D}_1 + \tilde{D}_{22}\check{D}_1)h_{21} : h_{12}, h_{21} \in \mathcal{C}^\infty\},\\
&d^2 = d^2_1 + d^2_2 = \check{D}_1\bar{\partial}_1 + \check{D}_2\bar{\partial}_2 + D_1\partial_1 + D_2\partial_2\\
&+\textstyle\sum_{i,j=1}^2(\check{D}_{ij}\bar{\partial}_i\bar{\partial}_j + D_{ij}\partial_i\partial_j + \tilde{D}_{ij}\partial_i\bar{\partial}_j + D^*_{ij}\bar{\partial}_i\partial_j),\\
&\\
&F_3 := \{(D_1D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)k_0 : k_0 \in \mathcal{C}^\infty\},\\
&d^3 = d^3_1 + d^3_2 = \check{D}_1\bar{\partial}_1 + \check{D}_2\bar{\partial}_2 + D_1\partial_1 + D_2\partial_2\\
&+\textstyle\sum_{i,j=1}^2(\check{D}_{ij}\bar{\partial}_i\bar{\partial}_j + D_{ij}\partial_i\partial_j + \tilde{D}_{ij}\partial_i\bar{\partial}_j + D^*_{ij}\bar{\partial}_i\partial_j),
\end{aligned}$$

and $\mathcal{C}^\infty$ denotes the sheaf of quaternionic valued infinitely differentiable functions on $\mathbb{H}^2$.

Proof. It follows from Propositions 3.3, 3.6 and 3.10. □

4 Hyperfunctions on 5-dimensional varieties in $\mathbb{H}^2$

In this section we finally use the results from the previous sections, as well as our early work [1], to construct explicitly the sheaf of hyperfunctions on the five dimensional variety L introduced in section 2. We then use the theory of megaforms to give a differential representation for these hyperfunctions.

To begin with we recall the resolution of the module associated to two Cauchy-Fueter operators, as we obtained it in [1], and some relevant notations.

Let R be the ring of polynomials $\mathbb{C}[\xi_1, \dots, \xi_8]$ and let $P(D)$ be the matrix associated to two Cauchy-Fueter operators

$$\begin{cases} \dfrac{\partial f}{\partial \bar{q}_1} = g_1 \\[2ex] \dfrac{\partial f}{\partial \bar{q}_2} = g_2 \end{cases} \tag{17}$$

where

$$P(D) = \begin{bmatrix} U_1(D) \\ U_2(D) \end{bmatrix}.$$

If we denote by P the Fourier transform of $P(D)$ and we denote by M the cokernel of the map induced by P^t, the Hilbert Syzygy Theorem states that there is a finite resolution of the module M. In [1] we show that a minimal resolution is

$$0 \longrightarrow R^4(-4) \overset{P_2^t}{\longrightarrow} R^8(-3) \overset{P_1^t}{\longrightarrow} R^8(-1) \overset{P^t}{\longrightarrow} R^4 \longrightarrow M \longrightarrow 0, \tag{18}$$

where the matrix P, P_1, P_2 can be written, by taking the real components of the matrices. With an abuse of notation, since we write them in quaternionic form we have:

$$P^t = \begin{bmatrix}\bar{q}_1 & \bar{q}_2\end{bmatrix} \qquad P_1^t = \begin{bmatrix}-\bar{q}_2 q_2 & q_1\bar{q}_2 \\ q_2\bar{q}_1 & -\bar{q}_1 q_1\end{bmatrix}, \qquad P_2^t = \begin{bmatrix} q_1 \\ q_2\end{bmatrix},$$

respectively. In addition, in [1] we show the following result showing that the flabby dimension of the sheaf of regular functions in two quaternionic variables equals 3:

Theorem 4.1. *If M is as above, then we have*

$$\mathrm{Ext}^i(M,R) = 0, \qquad for \quad i = 0,1,2$$

and

$$\mathrm{Ext}^3(M,R) \neq 0.$$

Our next result, see [1] and [6], follows from Theorem 4.1 and the ellipticity of the operator $P(D)$:

Theorem 4.2. *Let K be any bounded convex set, then*

$$H^j_K(\mathbb{H}^2,\mathcal{R}) = 0 \qquad j \neq 3$$

and

$$H^3_K(\mathbb{H}^2,\mathcal{R}) \cong [\tilde{\mathcal{R}}(K)]'$$

where $\tilde{\mathcal{R}}$ denotes the sheaf of anti-regular functions of two quaternionic variables.

At the same time, we have computed, in section 3, a resolution of the sheaf of regular functions in two (quaternionic) variables, and we have demonstrated how this resolution (which we have called a Dolbeault-like resolution) is equivalent to the Hilbert Syzygy resolution. As a consequence of the previous discussion, we give the following definition.

Definition 4.3. *Let L be an initial variety for the Cauchy-Fueter system in two variables as described in section 2. The presheaf associated to the pair $(U\cap L, H^3_L(U,\mathcal{R}))$, with U an open set in $\mathbb{H}^2$, is a sheaf on L which we will call sheaf of $\mathbb{H}$-hyperfunctions and we denote it by $\mathcal{B}_{\mathbb{H}^2,L}$.*

Remark 4.4. There are several delicate issues concerned with this definition, and its relevance. First, one would need to show that $(U\cap L, H^3_L(U,\mathcal{R}))$ is in fact a sheaf, and not just a presheaf. This is actually a consequence of the cohomology vanishings which we have proved in [1] (but see also our more detailed exposition in [6]). Also, the study of the cohomological properties of the sheaf $\mathcal{R}$ shows that the sheaf of $\mathbb{H}$-hyperfunctions is a flabby sheaf.

Our next step is to try to find a better way to characterize $\mathbb{H}$-hyperfunctions, by using the notion of megaforms. To begin with, we observe that it is immediate to write a fine resolution for the sheaf $\mathcal{R}$ of regular functions in two quaternionic variables, in fact

$$0 \longrightarrow \mathcal{R} \hookrightarrow (\mathcal{C}^\infty)^4 \xrightarrow{P(D)} (\mathcal{C}^\infty)^8 \xrightarrow{P_1(D)} (\mathcal{C}^\infty)^8 \xrightarrow{P_2(D)} (\mathcal{C}^\infty)^4 \longrightarrow 0 \tag{19}$$

is a resolution of $\mathcal{R}$. This is a consequence, see [6], of the minimal free resolution (18) and follows from the ellipticity of the Cauchy-Fueter operator.

Fine resolutions are particularly useful because they allow to easily compute the sheaf cohomologies, which would otherwise be rather unwieldy. In particular (see for example [10]), we have that, for any open set U in $\mathbb{H}^2$, the cohomology of U with values in $\mathcal{R}$ is given by the cohomology of the fine resolution (19). More precisely we have, for any open set as above,

$$H^j(U,\mathcal{R}) = \frac{\text{Ker } P_j(D)}{\text{Im } P_{j-1}(D)}, \tag{20}$$

where we have set $P_0(D) = P(D)$. An important consequence is the following cohomology vanishing theorem for convex open sets in $\mathbb{H}^2$.

Theorem 4.5. *Let U be an open convex set in $\mathbb{H}^2$. Then, for any $j \geq 1$, we have*

$$H^j(U,\mathcal{R}) = 0.$$

Proof. For $j \geq 3$ the statement is a consequence of the fact that the flabby dimension of the sheaf $\mathcal{R}$ is 3, see [3] and Theorem 4.1, so the statement holds for any open set U, not only for open convex sets. For $j = 1, 2$ we use Theorem 4.1 which shows that the kernel of $P_j(D)$ is the space of (pair of) functions satisfying the compatibility conditions for the system $P_{j-1}(D)$. Even though this is not sufficient, in general, to guarantee that these functions are images of functions by $P_{j-1}(D)$, this is however the case whenever we are considering a convex open set. In view of (20) this concludes the proof. □

We are now ready to show that the sheaf of quaternionic hyperfunctions in two variables is isomorphic to a sheaf of 2−megaforms. Indeed, we have the following theorem:

Theorem 4.6. *Let U be any open convex set in $\mathbb{H}^2$ and let L be a five-dimensional initial variety for the Cauchy-Fueter system. Then $\mathcal{B}_{\mathbb{H}^2,L}(U \cap L) \cong H^2(U \setminus L, \mathcal{R})$.*

Proof. Consider the relative cohomology exact sequence associated to the sheaf $\mathcal{R}$ and to the pair of open sets $(U, U \setminus L)$:

$$\ldots \to H^2(U,\mathcal{R}) \to H^2(U \setminus L,\mathcal{R}) \to H^3_L(U,\mathcal{R}) \to H^3(U,\mathcal{R}) \to \ldots.$$

Since U is a convex set, the previous theorem shows that $H^2(U,\mathcal{R}) = H^3(U,\mathcal{R}) = 0$, and the result follows immediately.

□

Remark 4.7. Note now that the open set $U \setminus L$ is not convex, and therefore its $\mathcal{R}$ cohomology does not vanish. Moreover, in an equivalent way, one can compute the cohomology using (16):

$$H^j(U \setminus L,\mathcal{R}) = \frac{\text{Ker } d^j}{\text{Im } d^{j-1}}.$$

So, putting $j = 2$, we can identify any quaternionic hyperfunction with an equivalence class of 2-megaforms. By recalling the characterization we have for the spaces F_2 and F_1, and by identifying coefficients in the natural order, we can finally say that a quaternionic hyperfunction f on a convex open set U can be identified with an equivalence class of pairs (f_1, f_2) of quaternionic valued, infinitely differentiable functions on $U \setminus L$, such that $\partial_1 f_1 + \partial_2 f_2 = 0$. Two pairs (f_1, f_2) and (g_1, g_2) are equivalent if there exists another pair (α_1, α_2) such that

$$\begin{cases} \Delta_1\alpha_2 - \bar\partial_2\partial_1\alpha_1 = f_1 - g_1 \\ \Delta_2\alpha_1 - \bar\partial_1\partial_2\alpha_2 = f_2 - g_2. \end{cases}$$

5 The case of Dirac operators

In this section we generalize the ideas of the previous sections to the case of Dirac operators. Let $\mathbb{R}^m$ be the real Euclidean space and define $\mathbb{R}_m$ as the real Clifford algebra generated by the orthogonal basis elements $\{e_1, \ldots, e_m\}$ together with the defining relations of the form

$$e_j e_k + e_k e_j = -2\delta_{jk}.$$

Let $\underline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ be a 1-vector variable (often written as $\underline{x} = \sum_{\ell=1}^m x_\ell e_\ell$) and let $f(\underline{x})$ be an $\mathbb{R}_m$–valued function. The Dirac operator is defined by

$$\partial_{\underline{x}} : \ f(\underline{x}) \to \partial_{\underline{x}} f(\underline{x}) = \sum_{j=1}^m e_j \partial_{x_j} f(\underline{x})$$

and the solutions to the equation $\partial_{\underline{x}} f(\underline{x}) = 0$ are called monogenic functions.

Let us consider functions of several vector variables $f : (\mathbb{R}^m)^k \longrightarrow \mathbb{R}_m$. The vector variables will be denoted by

$$\underline{x}_1, \ldots, \underline{x}_k, \qquad \underline{x}_j = \sum_{\ell=1}^m x_{j\ell} e_\ell, \quad j = 1, \ldots, k,$$

and the functions in the kernel of the corresponding Dirac operator $\partial_{\underline{x}_j}$, $j = 1, ..., k$ will be called monogenic in the given variables. When treating the Dirac equation as a system of 2^m real equations, each Dirac operator $\partial_{\underline{x}_j}$ will be associated to a matrix $U_j(D)$ while the system in k Dirac equations will be represented by a matrix $P(D)$. The megaforms associated to the complex of k Dirac operators have been studied, for $k = 2, 3$ in [15]. We have the following result, in which P will denote the Fourier transform of $P(D)$:

Theorem 5.1. *When $m > 2$, monogenic functions $\mathcal{M}$ in two variables can be embedded in the following Dolbeault–like complex:*

$$0 \to \mathcal{M} \hookrightarrow F_0 \stackrel{d^0}{\to} F_1 \stackrel{d^1}{\to} F_2 \stackrel{d^2}{\to} F_3 \stackrel{d^3}{\to} 0 \tag{21}$$

which is the complex corresponding to the resolution of the module $M = R/P^t$, where $R = \mathbb{C}[x_{11}, \ldots, x_{1m}, x_{21}, \ldots, x_{2m}]$

$$0 \longrightarrow R^{2^m}(-4) \longrightarrow R^{2^{m+1}}(-3) \longrightarrow R^{2^{m+1}}(-1) \stackrel{P^t}{\longrightarrow} R^{2^m} \longrightarrow M \to 0.$$

Remark 5.2. When $m = 2$, the two Dirac operators commute and the Dirac complex is (see [6])

$$0 \longrightarrow R^4(-2) \longrightarrow R^8(-1) \longrightarrow R^4 \longrightarrow M \to 0.$$

This corresponds to the usual Koszul complex, and as such can be dealt with in a simple and direct way.

In order to identify initial varieties for the Dirac system, we denote by $U_{\underline{x}}$ the 2^m by 2^m matrix corresponding to the multiplication by a vector variable $\underline{x} = \sum_{i=1}^m x_i e_i$ in the Clifford algebra $\mathbb{R}_m$. We can give an explicit representation of the matrix $U_{\underline{x}}$ as follows. Let $\underline{y}$ be any element in the Clifford algebra $\mathbb{R}_m$ and let

$$\underline{y} = \sum_{|A| \le m} y_A e_A, \quad y_A \in \mathbb{R}$$

be the representation of $\underline{y}$ with respect to the basis of multivectors

$$\{1,\ e_A = e_{a_1} \cdot \dots \cdot e_{a_t} \mid 1 \leq a_1 < \ldots < a_t \leq m,\ |A| \leq m\}.$$

Then the multiplication by $\underline{x}$ has a representation with respect of such basis given by

$$\underline{x}\,\underline{y} = \sum_{i=1}^{m} \sum_{|A|\leq m} x_i\, y_A\, e_i\, e_A = \sum_{i=1}^{m} \sum_{\substack{|A|\leq m \\ e_B = e_i e_A}} x_i\, y_A\, e_B.$$

Therefore the matrix $U_{\underline{x}}$ has elements u_{AB} given by

$$u_{AB} = \begin{cases} (-1)^N x_i & \text{if } e_i e_A = e_B,\ N = |A| + |B| + i + 1 \\ \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate to check that the diagonal elements u_{AA} are zero and that the matrix is skew-symmetric. Given two multi-indices A and B, there exists at most one basis vector e_i such that $e_i e_A = e_B$, and if we fix one of the multi-indices, say A, only n values of B are such that the entry u_{AB} is non zero, and they correspond to the m possible choices of e_i. From this argument we can conclude that each row (and each column) of U contains x_i, for each value of i between 1 and m, exactly once. In other words, the matrix associated to the multiplication by $\underline{x}$ can be written putting on its columns the vectors $\underline{x}e_A$, where A varies over all the possible subsets of $\{1, 2, \ldots, m\}$ and where we have set $e_\emptyset = 1$. We also have the following immediate result:

Proposition 5.3. *The determinant of the matrix $U_{\underline{x}}$ is $(x_1^2 + \ldots + x_m^2)^{2^{m-1}}$.*

Proof. First we note that since $U_{\underline{x}}$ represents the multiplication by $\underline{x}$, the matrix $U_{\underline{x}}^2$ represents multiplication by $\underline{x}^2$. But $\underline{x}^2 = -(x_1^2 + \cdots + x_m^2)$ and therefore the matrix which represents its multiplication is a diagonal matrix with $-(x_1^2 + \cdots + x_m^2)$ on each element of the diagonal. As a result, $\det(U_{\underline{x}}^2) = (x_1^2 + \ldots + x_m^2)^{2^m}$. By Binet's theorem, we obtain the result. □

We now consider the characteristic variety associated to the operator $P(D)$, corresponding to two Dirac operators. For sake of simplicity, we will denote the two operators by $\partial_{\underline{x}}$ and $\partial_{\underline{y}}$, where $\underline{x} = (x_1, \ldots, x_m)$, $\underline{y} = (y_1, \ldots, y_m)$ belong to $\mathbb{R}^m$. Before we describe the characteristic variety associated to the operator $P(D)$, we need some preliminary results on the Gröbner Basis of the module associated to P. We follow here an algebraic argument that has already been exploited in several papers on the algebraic analysis of operators, see e.g. [2].

Lemma 5.4. *Let $R = \mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_m]$, $m \geq 3$ be a ring of polynomials and let $P^t = [-U_{\underline{x}} \ \ -U_{\underline{y}}]$ be the transpose of the matrix associated to two Dirac operators. Then a degree reverse lexicographic reduced* Gröbner Basis *for the R–module $\langle P^t \rangle$ associated to P^t is given by the rows of P together with the rows of the matrix*

$$B_{\underline{x}\underline{y}} := U_{\underline{x}} U_{\underline{y}} - U_{\underline{y}} U_{\underline{x}}.$$

Proof. Explicit computations for the case of $m = 3$ can be carried on with CoCoA. Using the function `Coala.CliffordMat` available with the CoAlA package [8], it is possible to define the desired matrices $U_{\underline{x}}$ and $U_{\underline{y}}$ for virtually any dimension. Computations then shows that the reduced Gröbner Basis for the module $\langle P^t \rangle$ contains, besides the rows of the two matrices $U_{\underline{x}}$ and $U_{\underline{y}}$, the rows of the matrix

$$B'_{\underline{x}\underline{y}} := E_1 U_{\underline{x}} U_{\underline{y}} - E_1 U_{\underline{y}} U_{\underline{x}} = E_1 B_{\underline{x}\underline{y}},$$

where the matrix E_1 is the non degenerate matrix representing the multiplication by e_1 in $\mathbb{R}_m$. Up to signs, the multiplication by a vector is simply the automorphism given by a permutation of rows, so the rows of $B_{\underline{xy}}$ coincide with the rows of $B'_{\underline{xy}}$. For $m \geq 3$ the syzygies of the Dirac system are only radial (see for example [6]), i.e., they can be described only in terms of the two matrices $U_{\underline{x}}$ and $U_{\underline{y}}$. We conclude that the Gröbner Basis for our module is fully described by the case $m = 3$ because if there was a dependency on m this would be reflected in the way the syzygy are constructed, contradicting the radiality of the complex. □

The next step is to give an explicit expression for the leading term module $LT\langle P^t \rangle$ of the module associated to P^t and deduce a formula for the dimension of the the module $\mathrm{Coker}(P^t)$ by constructing a maximal regular sequence in the quotient $R^{2^m}/LT\langle P^t \rangle$. This is done in the two following lemmas.

Lemma 5.5. *Let $m \geq 3$. The leading term module $LT\langle P^t \rangle$ of the module associated to P^t is generated by the set*

$$\{x_1 e_A,\, y_1 e_A,\, x_3 y_2 e_A \,|\, |A| \leq m\}.$$

Proof. Again the statement needs to be verified with CoCoA only for the radial case $m = 3$. The leading term module is clearly diagonal, i.e., it is of the form $\bigoplus_{|A| \leq m} I e_A$, where I is a monomial ideal in R, because any of the rows of the matrices $U_{\underline{x}}$, $U_{\underline{y}}$ and $B_{\underline{xy}}$ is obtained from the first row of the respective matrix by a multiplication by e_A, and those are all the elements of the Gröbner Basis described in Lemma 5.4. Hence it just suffices to calculate the ideal I and this is obviously generated by the first nonzero monomials of the first row of $U_{\underline{x}}$, $U_{\underline{y}}$ and $B_{\underline{xy}}$ respectively, i.e. $I = (x_1, y_1, x_3 y_2)$. □

Lemma 5.6. *Let $m \geq 3$ and let M^* be the quotient module $R^{2^m}/LT\langle P^t \rangle$ and let $\wp$ be the maximal ideal of R generated by all the indeterminates. Then a maximal M^*-regular sequence in $\wp$ is given by the elements of the set*

$$\{x_2, x_3, y_3\} \cup \{x_i, y_i \,|\, i > 3\}.$$

Proof. Given the description of $LT\langle P^t \rangle$ given in Lemma 5.5, it is clear that a maximal regular sequence in the quotient M^* is given by those indeterminates that are not in I. In particular, it is again sufficient to find the regular sequence for $m = 3$, since for greater values of m we can just add the elements x_i, y_i for $3 < i \leq m$. It is trivial to see that a M^*-sequence for $m = 3$ is (x_2, x_3, y_3) and that this is maximal. □

Proposition 5.7. *The dimension of the characteristic variety associated to two Dirac operators in m real variables, with $m \geq 3$, is $2m - 3$.*

Proof. The geometric dimension of the algebraic variety described by the annihilation of the maximal minors of P is the Krull dimension of the module M^*. The latter is the same as the dimension of the quotient R/I defined in the previous Lemma, because the Hilbert series of M^* is just 2^m times the Hilbert series of R/I. The ideal I is Cohen-Macaulay so its dimension equals its depth. The M^*-sequence constructed in Lemma 5.6 is obviously also an I-sequence and it is maximal so the statement follows. □

As we have mentioned before, the characteristic variety is the set of points in $\mathbb{C}^{2m}$ where the rank of the $2^{m+1} \times 2^m$ complexified matrix P is less than the maximum: we are now ready to describe it. From now on, we will denote, as usual, by $\underline{x}$, $\underline{y}$ the original variables and by $\underline{z}$, $\underline{w}$ the corresponding dual variables.

Theorem 5.8. *The characteristic variety of the operator $P(D)$ associated to two Dirac operators is the set*

$$V = \{(\underline{z}, \underline{w}) \in \mathbb{C}^m \times \mathbb{C}^m \ : \ \underline{z}^2 = \underline{w}^2 = 0, \ \underline{z} \cdot \underline{w} = 0\}.$$

Proof. First we show that if $(\underline{z}, \underline{w})$ belongs to the characteristic variety then $(\underline{z}, \underline{w}) \in V$. To do so we note that if $(\underline{z}, \underline{w})$ belongs to the characteristic variety then the rank of the matrix

$$P = \begin{bmatrix} U_{\underline{z}} \\ U_{\underline{w}} \end{bmatrix}$$

is less than 2^m. This implies that the determinants of all the minors of order 2^m vanish. In particular the determinant of the matrices $U_{\underline{z}}$, $U_{\underline{w}}$ vanish. By proposition 5.3 this shows that $\underline{z}^2 = \underline{w}^2 = 0$. The vanishing of the determinants of $U_{\underline{z}}$ and $U_{\underline{w}}$ implies that

$$\sum a_A(z)\underline{z}e_A = 0 \qquad \text{and} \qquad \sum a_A(w)\underline{w}e_A = 0,$$

for suitable coefficients $a_A(z)$ and $a_A(w)$. As a consequence we have that also $\det(U_{\underline{z}+\underline{w}})$ vanishes and so $(\underline{z} + \underline{w})^2 = 0$ and we get $\underline{z}\underline{w} + \underline{w}\underline{z} = 0$ which is equivalent to $\underline{z} \cdot \underline{w} = 0$.

The converse inclusion follows from Proposition 5.7. In fact, V is irreducible and it has dimension $2m - 3$ because it is complete intersection, so the characteristic variety cannot be strictly included in V having the same dimension. □

Let us consider the subspace L of $\mathbb{R}^{2m}$ corresponding to the variables $\zeta'' = (x_1, \dots, x_m, y_1, \dots, y_{m-3})$, and let $\zeta' = (y_{m-2}, y_{m-1}, y_m)$. Let us denote the dual variables by $(z_1, \dots, z_m, w_1, \dots, w_{m-3})$ and $\zeta' = (w_{m-2}, w_{m-1}, w_m)$ respectively.

Proposition 5.9. *The subspace L is an initial variety for the Cauchy problem*

$$\begin{cases} \partial_{\underline{x}} f = 0 \\ \partial_{\underline{y}} f = 0 \\ f|_L = g. \end{cases} \tag{22}$$

Proof. Considering on the equality $w_1^2 + \dots + w_m^2 = 0$ and reasoning as in the proof of Theorem 2.3 we get that the Dirac system in two variables is weakly hypoelliptic in ζ'. The statement follows by Theorem 2.2. □

Definition 5.10. *Let L be an initial variety for the Dirac system in two variables. The presheaf associated to the pair $(U, H^3_L(\mathbb{H}^2, \mathcal{M}))$, with U an open set in L is a sheaf which we call sheaf of $\mathbb{H}$-hyperfunctions. In the sequel we will denote this sheaf by $\mathcal{B}_{\mathbb{R}^{2m},L}$.*

From (21) it follows that a fine resolution, analogue to (19), can be written also for the sheaf $\mathcal{M}$. Using this fine resolution we can compute, as in the Cauchy-Fueter case, the cohomology of an open set U with values in $\mathcal{M}$. As a consequence we immediately obtain the analogue of Theorem 4.5.

Theorem 5.11. *Let U be an open convex set in $\mathbb{R}^{2m}$. Then, for any $j \geq 1$, we have*

$$H^j(U, \mathcal{M}) = 0.$$

An immediate corollary is the following:

Corollary 5.12. *We have $\mathcal{B}_{\mathbb{R}^{2m},L}(U) \cong H^2(U \setminus L, \mathcal{M})$.*

Remark 5.13. As in the case of quaternionic hyperfunctions (see Remark 4.7), elements in $\mathcal{B}_{\mathbb{R}^{2m},L}(U)$ can be identified with suitable pairs of of $2-$megaforms in F_2.

References

[1] W.W. Adams, C.A. Berenstein, P. Loustaunau, I. Sabadini, D.C. Struppa, *Regular functions of several quaternionic variables and the Cauchy–Fueter complex*, J. Geom. Anal., **9** (1999), 1–16.

[2] W.W. Adams, P. Loustaunau, *Analysis of the module determining the properties of regular functions of several quaternionic variables*, Pacific J. Math., **196** (2000), 1–15.

[3] W.W. Adams, P. Loustaunau, V. Palamodov, D.C. Struppa, *Hartog's phenomenon for polyregular funtions and projective dimension of related modules*, Ann. Inst. Fourier, **47** (1997), 623–640.

[4] R. J. Baston, *Quaternionic Complexes*, J. Geom. Phys., **8** (1992), 29–52.

[5] J. Bureš, A. Damiano, I. Sabadini, *Explicit invariant resolutions for several Fueter operators*, J. Geom. Phys. **57** (2007), 765–775.

[6] F. Colombo, I. Sabadini, F. Sommen, D. C. Struppa, *Analysis of Dirac systems and computational algebra*, Progress in Mathematical Physics, Vol. 39, *Birkhäuser*, Boston, 2004.

[7] F. Colombo, V. Souček, D. C. Struppa, *Invariant resolutions for several Fueter operators*, J. Geom. Phys., , **56** (2006), 1175–1191.

[8] A. Damiano, S. Mannino, CoAlA , A web page for COmputational ALgebraic Analysis available at `http://www.tlc185.com/coala`.

[9] L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, New York, 1970.

[10] R. Godement, *Topologie Algébrique et Théorie des Faisceaux*, Hermann, Paris, 1958.

[11] A. Kaneko, *Introduction to Hyperfunctions*, Mathematics and its Applications, Kluwer, 1988.

[12] G. Kato, D. C. Struppa, *Fundamentals of Microlocal Algebraic Analysis*, Marcel Dekker, 1999.

[13] V.P. Palamodov, *Linear Differential Operators with Constant Coefficients*, Springer Verlag, New York 1970.

[14] R. Rocha-Chavez, M Shapiro, F. Sommen, *Integral theorems for functions and differential forms in* C^m, Research Notes in Math., Vol. 428, Chapman & Hall/CRC, (2001).

[15] I. Sabadini, F. Sommen, D.C. Struppa, *The Dirac complex on abstract vector variables: megaforms*, Exp. Math., **12** (2003), 351–364.

[16] I. Sabadini, F. Sommen, D.C. Struppa, P. Van Lancker, *Complexes of Dirac operators in Clifford algebras*, Math. Z., **239** (2002), 293–320.

[17] F. Sommen, *Microfunctions with values in a Clifford algebra II*, Sci. Papers College Arts Sci. Univ. Tokyo, **36** (1986), 15–37.