COMPUTATIONAL ALGEBRA FOR

ALGEBRAIC GEOMETRY AND STATISTICS

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Computation of Noetherian Operators

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Question 1

Given an ideal I in $R := \mathbb{C}[x_1, \ldots, x_n]$, how do we *test* if

$$f \in I$$

without using a division algorithm?

Question 2

How do we describe the algebraic set $V(I)$ so that we take into account *geometric multiplicities*?

Nullstellensatz:

$$\{\text{algebraic sets of } \mathbb{C}^n\}$$

$$\updownarrow$$

$$\{\text{radical ideals of } \mathbb{C}[x_1, \ldots, x_n]\}$$

Question 3

How do we *solve* a system of constant coefficients *partial differential equations*?

$$\begin{cases} \frac{\partial f}{\partial x} & = & -g \\ \frac{\partial f}{\partial y} & = & -\frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} & = & 0 \end{cases}$$

$\longrightarrow$ it could be done "by hand":

$f(x, y) = \frac{1}{2}ax^2 - bx + ay + c$

$g(x, y) = -ax + b$

Answer to Question i, $i = 1, 2, 3$:

Multiplicity Variety

Let V_j be algebraic varieties in $\mathbb{C}^n$ and let $\partial_j \in A_n = \mathbb{C}[x_1, \ldots, x_n, \partial x_1, \ldots, \partial x_n]$ be differential operators with polynomial coefficients, for $j = 1, ..., t$. Then we call

$$V = \{(V_1, \partial_1); (V_2, \partial_2); \ldots; (V_t, \partial_t)\}$$

a multiplicity variety.

Theorem Let I be an ideal of R. Then there exists a multiplicity variety V such that a polynomial f belongs to I if and only if $\partial_j f_{|V_j} = 0$ for every $j = 1, \ldots, t$.

Theorem

(Fundamental Principle of Ehrenpreis-Palamodov, 1960)

Let $p_1(D), \ldots, p_r(D)$ be linear constant coefficients partial differential operators in n variables. Then there are algebraic varieties $V_1, \ldots, V_t$ in $\mathbb{C}^n$ and differential operators $\partial_1, \ldots, \partial_t \in A_n$, such that every function $f \in \mathbb{C}^\infty(\mathbb{R}^n)$ satisfying

$$p_1(D)f = \ldots = p_r(D)f = 0$$

can be represented as

$$f(x) = \sum_{j=1}^{t} \int_{V_j} \partial_j(e^{ix\cdot z}) d\nu_j(z), \qquad (1)$$

for suitable Radon measures $d\nu_j$.

The above (1) is an integral representation of the solutions of the system of equations "given" by $I = (p_1, \ldots, p_r)$. It generalizes the well known **Euler's formula**

Problem

How do we actually compute a multiplicity Variety?

$$V = \{(V_1, \partial_1); (V_2, \partial_2); \ldots ; (V_t, \partial_t)\}$$

Primary Decomposition: $I = Q_1 \cap \cdots \cap Q_t$ gives

$$V(I) = V(Q_1) \cup \cdots \cup V(Q_t)$$

so we can define $\mathbf{V}_j = V(Q_j)$ and then we just look for the ∂_j's to attach to each irreducible component.

$\rightarrow$ The computation of a primary decomposition can be done on Singular and (soon) on CoCoA

How do we compute the Noetherian Operators?

Assume that I is primary. Some easy cases:

Case 1 (Euler's formula)
If $n = 1$ we have $I = (q(x)^\alpha)$ where $q(x) = x - a$ is irreducible. The operators in this case are simply

$$id, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x}, \ldots, \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}$$

Example
Given the ODE $\frac{\partial^\alpha f}{\partial z} = 0$ its symbol is $q(x) = x^\alpha$ and using the integral formula (1) we get $f(z) =$

$$= c_0(e^{xz})_{|0} + c_1 \frac{\partial(e^{xz})}{\partial x}_{|0} + \cdots + c_{\alpha-1} \frac{\partial^{\alpha-1}(e^{xz})}{\partial x^{\alpha-1}}_{|0} =$$

$$= c_0 + c_1 z + c_2 z^2 + \cdots + c_{\alpha-1} z^{\alpha-1}$$

Case 2 (Principal Ideals)

If $n \geq 1$ we have $I = (q(x_1, \ldots, x_n)^\alpha)$, $q(x_1, \ldots, x_n)$ irreducible. Again

$$id, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial^{\alpha-1}}{\partial x_1^{\alpha-1}}$$

a long as x_1 appears as a simple power in q (*normal position*)

Case 3 (Wave Equation)

$$\frac{\partial^2}{\partial z^2} u(z,t) - \frac{\partial^2}{\partial t^2} u(z,t) = 0$$

$$p(x,y) = x^2 - y^2 = (x+y)(x-y)$$

$$\mathbf{V} = \{(x+y=0, id); (x-y=0, id)\}$$

$$u(z,t) = F(z+t) + G(z-t)$$

Case 4 (Zerodimensional Ideals)

$$t = x_1^{i_1} \cdots x_n^{i_n}, \quad D(t) = D(i_1, \ldots, i_n) = \frac{1}{i_1! \cdots i_n!} \frac{\partial^{i_1 + \cdots + i_n}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$$

Space of differential operators:
$\mathcal{D} = \text{Span}_{\mathbb{C}}(\{D(t)\}) = \mathbb{C}[\partial x_1, \ldots, \partial x_n]$

Derivative-like morphisms σ_j:

$$\sigma_j \, D(i_1, \ldots, i_n) = \begin{cases} D(i_1, \ldots, i_j - 1, \ldots, i_n) & \text{if } i_j > 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Closed Subspaces
Consider $\mathcal{D}$ and extend σ_j to $\mathcal{D}$. A subspace U of $\mathcal{D}$ is *closed* if $\sigma_j(U) \subseteq U$, for all j

Correspondence $\Delta \longleftrightarrow \mathcal{I}$

Suppose that $V(I) = \{(0,\ldots,0)\}$:

$$\Delta(I) := \{L \in \mathcal{D} \,|\, L(f)(0,\ldots,0) = 0 \;\forall f \in I\}.$$

Similarly, we associate to each subset $U \subseteq \mathcal{D}$ an ideal

$$\mathcal{I}(U) := \{f \in R \,|\, L(f)(0,\ldots,0) = 0 \;\forall L \in U\}$$

Theorem Let $\mathfrak{m} = (x_1,\ldots,x_n)$. There is a **one to one** correspondence:

$$\{\mathfrak{m} - \text{primary ideals in } R\} \underset{\mathcal{I}}{\overset{\Delta}{\rightleftarrows}} \{\text{closed subspaces of } \mathcal{D}\}$$

so that $I = \mathcal{I}\Delta(U)$ and $U = \Delta\mathcal{I}(I)$.

For a zerodimensional $\mathfrak{m}$-primary ideal of R whose multiplicity is μ, we have that

$$\dim_{\mathbb{C}}(\Delta(I)) = \mu$$

.

Consequences:

Consider a primary zerodimensional ideal I centered at the origin with multiplicity μ:

- the noetherian operators associated to I are exactly μ

- the identity $id_{\mathcal{D}}$ is always a noetherian operator

- the maximum degree of an operator is $\mu - 1$

These facts and the use of Gröbner Bases techniques lead to an algorithm that can be easily implemented (CoCoA)

Algorithm

input: a Gröbner Basis $\mathcal{G}$ of I
output: $\Delta(I) = \{L_\beta\}$
1) Multiplicity of $\mu(I) = \dim_{\mathbb{C}}(R/I)$
2) Taylor expansion at the origin of a polynomial $h \in R$:

$$T_{\mu-1}h(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{N}^n,\, |\alpha| \leq \mu-1} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

3) Normal Form:

$$\mathrm{NF}_{\mathcal{G}}\, T_{\mu-1}h(x_1, \ldots, x_n) = \sum_\beta d_\beta x_1^{\beta_1} \cdots x_n^{\beta_n}$$

and find scalars $a_{\beta\alpha} \in \mathbb{C}$ such that

$$d_\beta = \sum_\alpha a_{\beta\alpha} c_\alpha$$

4) For each β such that $d_\beta \neq 0$, return the operator

$$L_\beta = \sum_\alpha a_{\beta\alpha} \frac{1}{\alpha!} \partial x_1{}^{\alpha_1} \cdots \partial x_n{}^{\alpha_n}$$

Example

$I = (y^2, x^2 - y) \subset \mathbb{C}[x, y]$,

1) Multiplicity $\mu = 4$

2) Taylor

$$T_3h(x,y) = c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 +$$

$$+ c_{30}x^3 + c_{21}x^2y + c_{12}xy^2 + c_{03}y^3$$

3) Normal Form

$$T_3h(x,y) = [c_{00}] + [c_{10}]x + [c_{01} + c_{20}]y + [c_{11} + c_{30}]xy$$

4) $[c_{00}] \rightarrow D(0,0) = 1$

$[c_{10}] \rightarrow D(1,0) = \partial x$

$[c_{01} + c_{20}] \rightarrow D(0,1) + D(2,0) = \partial y + \frac{1}{2}\partial x^2$

$[c_{11} + c_{30}] \rightarrow D(1,1) + D(3,0) = \partial xy + \frac{1}{6}\partial x^3$

Pros and Conts

$\longrightarrow$ polynomial complexity as MMM's algorithm

$\longrightarrow$ easy to implement on a computer

$\longrightarrow$ does not require linear solvers

$\longrightarrow$ needs a primary decomposition and a change of coordinates before it can be performed

Improvements and future work

- the same algorithm can be adapted to zerodimensional modules (CoCoA)

- using Noether Normalization it is possible to treat higher dimensional cases
The Noetherian Operators in this case belong to $\mathbb{C}[x_1, \ldots, x_d, \partial x_{d+1}, \ldots, \partial x_n]$

- we look for a better bound for the Taylor expansion, other than $\mu - 1$

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