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QUATERNIONIC-LIKE COMPLEXES FOR HIGHER-SPIN DIRAC OPERATORS

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OUTLINE

1. Higher-spin operators in real dimension 3

2. Examples of computations, results on free resolutions

3. Dirac operator in dimension m: some problems

4. Massless fields operators in real dimension 4

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⇒ **1.** Higher-spin operators in real dimension 3

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REFERENCES

[1] Y.Homma, **The Higher spin Dirac operators on 3–dimensional manifolds**. Tokyo J. Math. **24** (2001), no. 2, 579–596.

[2] J. Bureš, F. Sommen, V. Souček, P. Van Lancker, **Rarita-Schwinger type operators in Clifford analysis**, J. Funct. Anal. **185** (2001), no. 2, 425–455

DEFINITIONS

(ρ_j, V_j) irreducible representation of $SU(2)$, $(\rho_2, V_2) \simeq (ad, \mathbb{R}^3 \otimes \mathbb{C})$,

$$(\rho_j, V_j) \otimes (\rho_2, V_2) \simeq (\rho_{j+2}, V_{j+2}) \oplus (\rho_j, V_j) \oplus (\rho_{j-2}, V_{j-2})$$

$$\nabla^S : S_j \longrightarrow S_j \otimes T^*(M)$$

$S_j \otimes T^*M \simeq S_{j+2} \oplus S_j \oplus S_{j-2}$ $\quad \pi_+, \pi_0, \pi_-$ projections

$$\mathcal{D}_j := \pi_0 \circ \nabla^S : \; S_j \longrightarrow S_j$$

Let us focus on the **flat version** of the operator...

FLAT REPRESENTATION

Y.Homma [1] and Mathematica®

$\{e_1, e_2, e_3\}$ a basis of $\mathbb{R}^3$ $\{z^k\}_{k=0..j}$ a basis of V_j

Higher-spin Dirac $D_j : \Gamma(V_j) \longrightarrow \Gamma(V_j)$ $D_j = \sum_{i=1}^{3} \rho_j^0(e_i) \cdot \nabla_{e_i}$

Clifford morphism

$$\begin{cases} \rho_j^0(\frac{e_1}{2})z^k = i(k - \frac{j}{2})z^k \\ \rho_j^0(\frac{e_2}{2} + i\frac{e_3}{2})z^k = -kz^{k-1} \\ \rho_j^0(\frac{e_2}{2} - i\frac{e_3}{2})z^k = (j-k)z^{k+1}. \end{cases}$$

SYMBOL MATRIX

- $$\nabla_{e_i} = \frac{\partial}{\partial x_i} \longrightarrow x_i$$

- change of coordinates $x = -ix_1 \quad y = x_2 + ix_3 \quad z = -x_2 + ix_3$

- Coordinate ring: $R := \mathbb{C}[x, y, z]$

- The **symbol** of D_j is a matrix $P_j \in \mathrm{Mat}_{j+1}(R)$

Expression from Homma [1]: $P_1 := \begin{pmatrix} -ix_1 & -x_2 + ix_3 \\ x_2 + ix_3 & ix_1 \end{pmatrix}$

After a change of coordinates: $P_1 = \begin{pmatrix} -x & z \\ y & x \end{pmatrix}$

This is a complex version of Moisil-Theodorescu on $\mathcal{C}^\infty(\mathbb{R}^3, \mathbb{H})$

Expression from Homma [1]:

$$P_2 := \begin{pmatrix} -2ix_1 & -x_2 + ix_3 & 0 \\ 2x_2 + 2ix_3 & 0 & -2x_2 + 2ix_3 \\ 0 & x_2 + ix_3 & 2ix_1 \end{pmatrix}$$

After a change of coordinates: $P_2 = \begin{pmatrix} 2x & z & 0 \\ 2y & 0 & 2z \\ 0 & y & -2x \end{pmatrix}$

This is NOT ELLIPTIC and in fact it behaves differently.

Expression from Homma [1]:

$$P_3 := \begin{pmatrix} -3ix_1 & -x_2 + ix_3 & 0 & 0 \\ 3x_2 + 3ix_3 & -ix_1 & -2x_2 + 2ix_3 & 0 \\ 0 & 2x_2 + 2ix_3 & ix_1 & -3x_2 + 3ix_3 \\ 0 & 0 & x_2 + ix_3 & 3ix_1 \end{pmatrix}$$

After a change of coordinates: $P_3 = \begin{pmatrix} 3x & z & 0 & 0 \\ 3y & x & 2z & 0 \\ 0 & 2y & -x & 3z \\ 0 & 0 & y & -3x \end{pmatrix}$

This is the simplest elliptic higher-spin version of Dirac.

HIGHER-SPIN DIRAC

Spin $j/2$

$$P_j = \begin{pmatrix} \mathbf{j}x & z & 0 & 0 & \dots & 0 & 0 \\ \mathbf{j}y & (\mathbf{j}-\mathbf{2})x & \mathbf{2}z & 0 & \dots & 0 & 0 \\ 0 & (\mathbf{j}-\mathbf{1})y & (\mathbf{j}-\mathbf{4})x & \mathbf{3}z & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathbf{3}y & (\mathbf{4}-\mathbf{j})x & (\mathbf{j}-\mathbf{1})z & 0 \\ 0 & 0 & \dots & 0 & \mathbf{2}y & (\mathbf{2}-\mathbf{j})x & \mathbf{j}z \\ 0 & 0 & \dots & 0 & 0 & y & -\mathbf{j}x \end{pmatrix}$$

$$\mathrm{Det}(P_j) = \prod_{k=0}^{j}(j-2k)\cdot(x^2+yz)^{\frac{j+1}{2}} = \gamma_j \cdot (x_1^2+x_2^2+x_3^2)^{\frac{j+1}{2}}$$

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REFERENCES

[3] F. Colombo, I. Sabadini, F. Sommen, D. C. Struppa, **Analysis of Dirac systems and computational algebra**, Progress in Mathematical Physics, 39, *Birkhäuser*, Boston, 2004.

[4] CoCoA, A software package for **Co**mputations in **Co**mmutative **A**lgebra, freely available at `http://cocoa.dima.unige.it`

• We want to study algebraic invariants of $\boxed{R^{j+1}/M_j}$
where $M = \mathrm{Im}(P^T)$, $P^T = [P^T_{j1}, \dots, P^T_{jn}]$

$$\updownarrow$$

• This captures the algebraic nature of $\boxed{P(D)f = g}$
with $f, g \in \mathcal{C}^\infty$, $D = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$

• We then look for a free resolution of the R module:

$$0 \longrightarrow R^{\beta_t} \longrightarrow \dots \longrightarrow R^{\beta_3} \longrightarrow R^{\beta_2} \longrightarrow R^{n(j+1)} \xrightarrow{P^T} R^{j+1} \longrightarrow 0$$

DIRAC

CoCoA results, $j = 1$, $m = 3$

$n = 1$

$$0 \longrightarrow R^{\mathbf{2}} \xrightarrow{P_1^T} R^{\mathbf{2}} \longrightarrow 0$$

$n = 2$

$$0 \longrightarrow R^{\mathbf{2}} \longrightarrow R^{\mathbf{4}} \longrightarrow R^{\mathbf{4}} \xrightarrow{P_2^T} R^{\mathbf{2}} \longrightarrow 0$$

$n = 3$

$$0 \longrightarrow R^{\mathbf{4}} \longrightarrow R^{\mathbf{18}} \longrightarrow R^{\mathbf{30}} \longrightarrow R^{\mathbf{20}} \longrightarrow R^{\mathbf{6}} \xrightarrow{P_3^T} R^{\mathbf{2}} \longrightarrow 0$$

$\dots$

$n \geq 3$

$$0 \longrightarrow R^{\mathbf{2(n-1)}} \longrightarrow \dots \longrightarrow R^{\binom{\mathbf{2n}}{\mathbf{3}}} \longrightarrow R^{\mathbf{2n}} \xrightarrow{P_n^T} R^{\mathbf{2}} \longrightarrow 0$$

RARITA-SCHWINGER

CoCoA results, $j = 3$, $m = 3$

$n = 1$

$$0 \longrightarrow R^{\mathbf{4}} \xrightarrow{P_1^T} R^{\mathbf{4}} \longrightarrow 0$$

$n = 2$

$$0 \longrightarrow R^{\mathbf{4}} \longrightarrow R^{\mathbf{8}} \longrightarrow R^{\mathbf{8}} \xrightarrow{P_2^T} R^{\mathbf{4}} \longrightarrow 0$$

$n = 3$

$$0 \longrightarrow R^{\mathbf{8}} \longrightarrow R^{\mathbf{36}} \longrightarrow R^{\mathbf{60}} \longrightarrow R^{\mathbf{40}} \longrightarrow R^{\mathbf{12}} \xrightarrow{P_3^T} R^{\mathbf{4}} \longrightarrow 0$$

$\ldots$

$n \geq 3$

$$0 \longrightarrow R^{\mathbf{4(n-1)}} \longrightarrow \ldots \longrightarrow R^{\mathbf{2\binom{2n}{3}}} \longrightarrow R^{\mathbf{4n}} \xrightarrow{P_n^T} R^{\mathbf{4}} \longrightarrow 0$$

HIGHER-SPIN CoCoA results, $j = 5$, $m = 3$

$n = 1$

$$0 \longrightarrow R^{\mathbf{6}} \xrightarrow{P_1^T} R^{\mathbf{6}} \longrightarrow 0$$

$n = 2$

$$0 \longrightarrow R^{\mathbf{6}} \longrightarrow R^{\mathbf{12}} \longrightarrow R^{\mathbf{12}} \xrightarrow{P_2^T} R^{\mathbf{6}} \longrightarrow 0$$

$n = 3$

$$0 \longrightarrow R^{\mathbf{12}} \longrightarrow R^{\mathbf{48}} \longrightarrow R^{\mathbf{90}} \longrightarrow R^{\mathbf{60}} \longrightarrow R^{\mathbf{18}} \xrightarrow{P_3^T} R^{\mathbf{6}} \longrightarrow 0$$

$$\dots$$

$n \geq 3$

$$0 \longrightarrow R^{\mathbf{6(n-1)}} \longrightarrow \dots \longrightarrow R^{\mathbf{4\binom{2n}{3}}} \longrightarrow R^{\mathbf{6n}} \xrightarrow{P_n^T} R^{\mathbf{6}} \longrightarrow 0$$

RESULTS

j odd, $m = 3$

• The resolution has length $2n-1$ and linearizes after the first syzygies (quadratic)

$$0 \longrightarrow R^{\beta_{2n-1}}_{(-2n)} \longrightarrow \ldots \longrightarrow R^{\beta_3}_{(-4)} \xrightarrow{Q_2^T} R^{\beta_2}_{(-3)} \xrightarrow{Q_1^T} R^{(j+1)n}_{(-1)} \xrightarrow{P^T} R^{j+1} \longrightarrow 0$$

$$\beta_d = n(\mathbf{j}+\mathbf{1})\binom{2n-1}{d}\frac{d-1}{d+1}, \quad d > 1.$$

• Key ingredient: standard Gröbner Basis techniques.

COMPLEX OF OPERATORS

applying results from [3]

• Let U be a convex open set of $\mathbb{R}^3$ and let $\mathcal{S} = \mathcal{C}^\infty(\mathbb{R}^{3n}, V_j)$ be the sheaves of smooth functions with values in the irreducible representation V_j, $j = 1, 3, 5, \ldots$. Let S^P be the kernel of $P(D)$ on S. Then the dual of the free resolution

$$0 \longrightarrow \mathcal{S}^P(U) \longrightarrow \mathcal{S}^{\beta_0} \xrightarrow{P(D)} \mathcal{S}(U)^{\beta_1} \xrightarrow{Q_1(D)} \mathcal{S}(U)^{\beta_2} \xrightarrow{Q_2(D)} \ldots \xrightarrow{Q_{2n-2}(D)} \mathcal{S}(U)^{\beta_{2n-1}} \longrightarrow 0$$

is an exact complex.

• Key ingredient: computation of $\mathrm{Ext}_R(R^{j+1}/M_j, R)$

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DEFINITION real dimension m

⋆ On smooth functions $f : \mathbb{R}^m \longrightarrow \mathcal{C}_m$ we have $\boxed{\partial = \sum_i e_i \frac{\partial}{\partial x_i}}$

⋆ Consider the system

$$\begin{cases} \partial_1 f = g_1 \\ \dots \\ \partial_n f = g_n \end{cases}$$

for $f : (\mathbb{R}^m)^n \longrightarrow \mathcal{C}_m$ and try to apply some algebraic methods...

⋆ Many questions still need to be answered (Souček, Franek, Sabadini)

PARTIAL ANSWERS

⋆ **Franek**: calculation of a complex using Verma modules:

- the complex is symmetric, for $m \geq 2n$
- syzygies are at most quadratic
- Betti numbers using representation theory

⋆ **Damiano, Sabadini**: calculation of a complex using Gröbner Bases

- for $R^{2^m}/M_{m,n}$ we have $\mathrm{LT}(M_{m,n})$ hence the Hilbert series
- the exceptional behavior for $m < 2n$ is clear from $\mathrm{LT}(M_{m,n})$
- exactness follows from the Cohen-Macaulyness of $M_{m,n}$

CoAlA : visit our webpage `www.tlc185.com/coala` for updates

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REFERENCES

[5] J. Bureš, V. Souček, **Complexes of invariant operators in several quaternionic variables**, to appear on Complex Variables: Theory and Applications

[6] F. Colombo, V. Souček, D.C. Struppa, **Invariant resolutions for several Fueter operators**, J. Geom. Phys. **56** (2006) 1175-1191

[7] J. Bureš, A. Damiano, I. Sabadini, **Explicit resolutions for the complex of several Fueter operators**, to appear on J. Geom. Phys.

INVARIANT OPERATORS

Theorem from [5]

The only invariant first order systems on $\mathbb{HP}^n$ made by n independent equations in n different quaternionic variables are

massless $\boxed{\nabla^{i,A}_{A'}\varphi_{A\dots D} = 0}$ $A\dots D \in \{0,1\}$, $i \in \{1\dots n\}$

twistor $\boxed{\nabla^{i,A}_{(A'}\varphi_{B'\dots D')} = 0}$ $A\dots D \in \{0,1\}$, $i \in \{1\dots n\}$

$$\nabla^{i,0}_{0'} = \partial x_i + \mathbf{i}\partial y_i, \qquad \nabla^{i,0}_{1'} = -\partial z_i - \mathbf{i}\partial t_i$$

$$\nabla^{i,1}_{0'} = \partial z_i - \mathbf{i}\partial t_i, \qquad \nabla^{i,1}_{1'} = \partial x_i - \mathbf{i}\partial y_i$$

$$R := \mathbb{C}[x_i, y_i, z_i, t_i]_{i=1\dots n}, \qquad \mathbb{H} = \mathbb{C} \oplus \mathbf{j}\mathbb{C}$$

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$$\nabla^{i,0}_{0'} = \partial x_i + \mathbf{i}\partial y_i, \qquad \nabla^{i,0}_{1'} = -\partial z_i - \mathbf{i}\partial t_i$$

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SYMBOL MATRIX

massless field operator

$$
(\mathrm{j}+1)\times 2\mathrm{j}:\quad
\left(\begin{array}{cccccc}
\nabla^{i,0}_{0'} & \nabla^{i,0}_{1'} & 0 & \dots & 0 & 0 \\
0 & \nabla^{i,0}_{0'} & \nabla^{i,0}_{1'} & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & \nabla^{i,0}_{0'} & \nabla^{i,0}_{1'} \\
\hline
\nabla^{i,1}_{0'} & \nabla^{i,1}_{1'} & 0 & \dots & 0 & 0 \\
0 & \nabla^{i,1}_{0'} & \nabla^{i,1}_{1'} & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & \nabla^{i,1}_{0'} & \nabla^{i,1}_{1'}
\end{array}\right)
$$