# Math 370 Computational Algebra - Notes

*When is a point on a variety?*

**Proposition.** Let $\mathbb{K}$ be a field, and lat $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_n]$ be the ring of multivariate polynomials. Consider a point in the affine space $\mathcal{P} = (p_1, \ldots, p_n) \in \mathbb{K}^n$ and an ideal $I \subset \mathcal{R}$. Denote with $\mathcal{V}(I)$ the variety associated to $I$. Then

$$\mathcal{P} \in \mathcal{V}(I) \quad \Leftrightarrow \quad I \subseteq \langle x_1 - p_1, \ldots, x_n - p_n \rangle.$$

*Proof.*
" $\Leftarrow$ " This is a special case of the lemma we proved in class, with $\mathcal{V}(J) = \mathcal{P}$ and $J = \langle x_1 - p_1, \ldots, x_n - p_n \rangle$.

" $\Rightarrow$ " Let $f \in I$ and $d = \delta(f)$. We want to show that $f$ is a combination of the polynomials $x_1 - p_1, \ldots, x_n - p_n$. From *multivariable calculus*, we know that the function $f(x_1, \ldots, x_n)$ can be expanded using Taylor's formula:

$$\begin{aligned}
f(x_1, \ldots, x_n) \quad &= \quad f(p_1, \ldots, p_n) + \\
&+ \quad \tfrac{\partial f(p_1,\ldots,p_n)}{\partial x_1}(x_1 - p_1) + \cdots + \tfrac{\partial f(p_1,\ldots,p_n)}{\partial x_n}(x_n - p_n) + \\
&+\tfrac{1}{2}\left( \tfrac{\partial^2 f(p_1,\ldots,p_n)}{\partial x_1^2}(x_1 - p_1)^2 + 2\tfrac{\partial^2 f(p_1,\ldots,p_n)}{\partial x_1 \partial x_2}(x_1 - p_1)(x_2 - p_2) + \cdots + \tfrac{\partial^2 f(p_1,\ldots,p_n)}{\partial x_n^2}(x_n - p_n)^2\right) + \\
&+ \quad \cdots + \\
&+ \quad \textstyle\sum_{|\alpha| = d} \tfrac{1}{\alpha!} \tfrac{\partial^{|\alpha|} f(p_1,\ldots,p_n)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x_1 - p_1)^{\alpha_1} \cdots (x_n - p_n)^{\alpha_1} + \\
&+ \quad r(x).
\end{aligned}$$

Observe the following:
- $f(p_1, \ldots, p_n) = 0$ because $\mathcal{P}$ is on the variety, so it is a common zero of all polynomials in $I$.
- All terms up to degree $d$ are in the ideal $I$ because they are all multiples of some $(x_i - p_i)$.
- (this is crucial) the remainder $r(x)$ satisfies the following property:

$$\lim_{(x_1,\ldots,x_n) \to (p_1,\ldots,p_n)} \frac{r(x)}{|x - p|^d} = 0.$$

Since $f$ is a polynomial, $r$ has to be a polynomial as well. The only polynomials who satisfy the above property are polynomials of degree at least $d + 1$ in $(x_i - p_i)$. However, since the degree of $f$ must be $d$, the only possible way for this to happen is that $r = 0$. $\square$

**Remark 1.** To see if $(p_1, \ldots, p_n)$ is a solution to the system

$$\begin{cases} f_1(x_1, \ldots, x_n) = 0 \\ f_2(x_1, \ldots, x_n) = 0 \\ \qquad \vdots \\ f_s(x_1, \ldots, x_n) = 0, \end{cases}$$

one could "plug in" the coordinates to see if it solves all equations, or check whether $\langle f_1, \ldots, f_s \rangle$ is contained in the ideal $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$. This amounts to testing membership of each $f_i$ to the ideal $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$.

**Remark 2.** Taylor's formula seems to hold only if $\mathbb{K} = \mathbb{R}$. It is actually a much more general fact that holds true for any (good...) ring of coefficients. If you are interested in this problem let me know, you can choose it as the final project for this class.